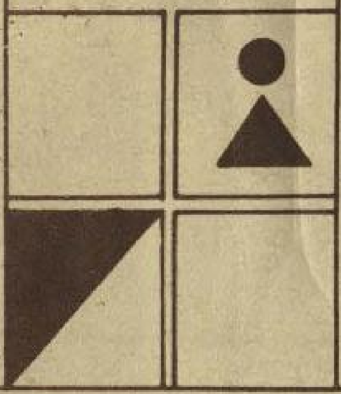
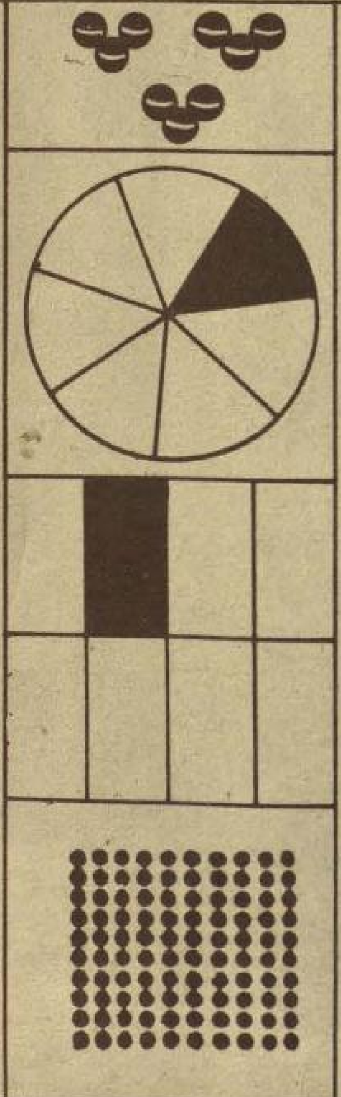


GENERAL MATHEMATICS
FOR CLASS IX.X



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ANTI-LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6.	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6775	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	6	8	11	13	15	17	19
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	15	17	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

ANTI-LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0 0 1	1 1 1	2 2 2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0 0 1	1 1 1	2 2 2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0 0 1	1 1 1	2 2 2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0 0 1	1 1 1	2 2 2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0 1 1	1 1 2	2 2 2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0 1 1	1 1 2	2 2 2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0 1 1	1 1 2	2 2 2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0 1 1	1 1 2	2 2 2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0 1 1	1 1 2	2 2 3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0 1 1	1 1 2	2 2 3
·10	1259	1262	1265	1268	1271	1274	1270	1279	1282	1285	0 1 1	1 1 2	2 2 3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0 1 1	1 2 2	2 2 3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0 1 1	1 2 2	2 2 3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0 1 1	1 2 2	2 3 3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0 1 1	1 2 2	2 3 3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0 1 1	1 2 2	2 3 3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0 1 1	1 2 2	2 3 3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0 1 1	1 2 2	2 3 3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0 1 1	1 2 2	2 3 3
·19	1549	1552	1556	1500	1563	1507	1570	1574	1578	1581	0 1 1	1 1 2	3 3 3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0 1 1	1 2 2	3 3 3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0 1 1	2 2 2	3 3 3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0 1 1	2 2 2	3 3 3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0 1 1	2 2 2	3 3 4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0 1 1	2 2 2	3 3 4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0 1 1	2 2 2	3 3 4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0 1 1	2 2 3	3 4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0 1 1	2 2 3	3 3 4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0 1 1	2 2 3	3 4 4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0 1 1	2 2 3	3 4 4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0 1 1	2 2 3	3 4 4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0 1 1	2 2 3	3 4 4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0 1 1	2 2 3	3 4 4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0 1 1	2 2 3	3 4 4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1 1 2	2 3 3	4 4 5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1 1 2	2 3 3	4 4 5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1 1 2	2 3 3	4 4 5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2385	2393	1 1 2	2 3 3	4 4 5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1 1 2	2 3 3	4 4 5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2500	1 1 2	2 3 3	4 5 5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1 1 2	2 3 4	4 5 5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1 1 2	2 3 4	4 5 5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1 1 2	2 3 4	4 5 6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1 1 2	3 3 4	4 5 6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1 1 2	3 3 4	4 5 6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1 1 2	3 3 4	5 5 6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1 1 2	3 3 4	5 5 6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1 1 2	3 3 4	5 5 6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1 1 2	3 4 4	5 6 6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1 1 2	3 4 4	5 6 6

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CHAPTER 1

NUMBER SYSTEMS

1.1 DECIMAL SYSTEM AND BASE

The system of number currently in use is the Decimal System because its base is ten which is of basic importance. It is easy to denote numbers taking 10 as base. We can also write numbers in expanded notation involving powers of 10, e.g.

$$23 = 20 + 3$$

$$= 2 \times 10^{1} + 3 \times 10^{0}$$

and

$$4170 = 4 \times 1000 + 1 \times 100 + 7 \times 10 + 0$$

$$= 4 \times 10^{3} + 1 \times 10^{2} + 7 \times 10^{1} + 9 \times 10^{0}$$

$$31.39 = 3 \times 10^{1} \times 1 \times 10^{0} + \frac{2}{10} + \frac{9}{100}$$

$$= 3 \times 10^{1} + 1 \times 10^{0} + 2 \times 10^{-1} + 9 \times 10^{-2}$$

The introduction of this system is a matter of chance. Since there are ten fingers in a person's hands and in the olden times the people used to count with the help of fingers, therefore, 10 was taken as a base for counting. The reality is that we can use any number as a base for counting. In some practical problems the systems other than the Decimal System have proved to be useful, e.g., Binary System, i.e., Base-2 System is used in computers.

We shall study here the systems of numbers with base five and base two.

---

* Zero power of every non-zero number equals 1.

1.2. BASE FIVE SYSTEM

You know that in order to represent numbers in Base 10, we use ten digits namely 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

In the system with base-five, we need only five digits which we take to be 0, 1, 2, 3, 4.

In order to convert a number in base-five, we write that number in expanded notation involving powers of five. The method is explained below with the help of examples.

Example 1

Convert 8 into a number of base-five.

Solution:

$$\begin{array}{c|l} 5 & 8 \\ \hline & 1-3 \end{array}$$

So $8 = 1 \times 5^1 + 3 \times 5^0$

$= (13)_5$

The word 'Five' written at the foot indicates that the number belongs to base-five system.

Note : 13_{five} is read as "one three with base five". We cannot read it as "thirteen with base-five".

Example 2

Convert 6720 into a number of base-five.

Solution:

$$\begin{array}{c|r} 5 & 6720 \\ \hline 5 & 1344-0 \\ \hline 5 & 268-4 \\ \hline 5 & 53-3 \\ \hline 5 & 10-3 \\ \hline & 2-0 \end{array}$$

So $6720 = 2 \times 5^5 + 0 \times 5^4 + 3 \times 5^3 + 3 \times 5^2 + 4 \times 5^1 + 0 \times 5^0$

$= (203340)_5$

Note: Here 6720 means 6720_{10} . While writing numbers in base-ten system we usually do not write 10 at the foot of a number.

We can easily convert numbers of base-five into numbers of base 10. Illustrative examples are given below:

Example 3

Write $(1023)_5$ in Decimal system.

Solution: $(1023)_5 = 1 \times 5^3 + 0 \times 5^2 + 2 \times 5^1 + 3 \times 5^0$

$= 125 + 0 + 10 + 3$

$= 138$

Example 4 Write $(234012)_5$ in Decimal system.

Solution:

$(234012)_5 = 2 \times 5^5 + 3 \times 5^4 + 4 \times 5^3 + 0 \times 5^2 + 1 \times 5^1 + 2 \times 5^0$

$= 2 \times 3125 + 3 \times 625 + 4 \times 125 + 0 + 5 + 2$

$= 6250 + 1875 + 500 + 5 + 2$

$= 8632$

An easier way of doing this sum will be clear from the following solution:

$2\,3\,4\,0\,1\,2_5$

$\underline{\times 5}$

$10 + 3 = 13$

$\underline{\times 5}$

$65 + 4 = 69$

$\underline{\times 5}$

$345 + 0 = 345$

$\underline{\times 5}$

$1725 + 1 = 1726$

$\underline{\times 5}$

$8630 + 2 = 8632$

EXERCISE NO. 1.1

Convert the following numbers into those of Decimal System:

(1) $(32)_5$ (2) $(201)_5$ (3) $(3000)_5$
(4) $(2341)_5$ (5) $(10243)_5$

Convert the following numbers into those of Base-five System:

(6) 23 (7) 39 (8) 76 (9) 112 (10) 324

1.3. BINARY SYSTEM OR BASE-TWO SYSTEM

Since the base in Binary System is two, therefore, we use only two digits, namely 0 and 1, in order to represent numbers in this system. The method for conversion of numbers from Decimal System to Base-two System and vice versa is illustrated below with the help of examples.

Example 1

Represent 21 in Base-two system

Solution:

2	21
2	10 – 1
2	5 – 0
2	2 – 1
	1 – 0

i.e. $21 = 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = (10101)$

Example 2

Represent $(10111001)_2$ in Decimal System.

Solution

$$(10111001)_2 = 1\times2^7+0\times2^6+1\times2^5+1\times2^4+1\times2^3+0\times2^2+0\times2^1+1\times2^0$$

$$= 128+0+32+16+8+0+0+1 = 185$$

EXERCISE NO. 1.2

Convert the following numbers into those of Base-Two System.

(1) 5 (2) 17 (3) 32 (4) 63 (5) 81

Convert the following numbers into those of Decimal System :

(6) $(101)_2$ (7) $(1110)_2$ (8) $(101010)_2$

(9) $(1111001)_2$ (10) $(1100001)_2$

1.4. ADDITION AND SUBTRACTION IN BASE-FIVE.

It has already been mentioned that we use ten digits (from 0 to 9) in the decimal system ; five digits (from 0 to 4) in the base-five system. So while adding in base-five system 1 is carried to the next place for every five.

e.g.

$$\begin{array}{r} 3_5 \\ +4_5 \\ \hline 12_5 \end{array}$$

Here the sum of 3_{five} and 4_{five} is 7 but in base-five system there is no 7 as such, so we think of 7 as 5 + 2. 2 takes the place of units and 1-five (i.e. 1×5^1) is carried to the next place. Similarly

$$\begin{array}{r} 1_5 \\ +4_5 \\ \hline 10_5 \end{array} \qquad \begin{array}{r} 2_5 \\ +4_5 \\ \hline 11_5 \end{array} \qquad \begin{array}{r} 3_5 \\ +3 \\ \hline 11_5 \end{array} \qquad \begin{array}{r} 4_5 \\ +4_5 \\ \hline 13_5 \end{array}$$

ADDITION TABLE IN BASE-FIVE SYSTEM

+	0	1	2	3	4
0	0	2	2	3	4
1	1	2	3	4	$(10)_5$

2	2	3	4	10_5	11_5
3	3	4	10_5	11_5	12_5
4	4	10_5	11_5	12_5	13_5

To illustrate the operation of addition further some more exampes are given below :

Example 1

Add 241_5 to 342_5

Solution

$$\begin{array}{r} 241_5 \\ +342_5 \\ \hline 1133_5 \end{array}$$

Example 2

Subtract 314_5 from 421_5.

Solution

$$\begin{array}{r} 421_5 \\ -314_5 \\ \hline 102_5 \end{array}$$

Explanation

4 cannot be subtracted from 1, therefore, we borrow 1 (which is one five in fact) from 2 on the left hand side. Five and one make six. Now subtracting 4 from six, we get 2 and so on.

Example 3

Subtract $(24344)_5$ from $(32012)_5$

Solution

$$\begin{array}{r} (32012)_5 \\ -(24344)_5 \\ \hline (2113)_5 \end{array}$$

EXERCISE NO. 1.3

Convert into base-five system and add:-

(1) $37 + 18$ (2) $101 + 34$

Find the sum and verify the answer by converting the numbers into Decimal system:

(3) $14_5 + 32_5$ (4) $101_5 + 20_5$

(5) $3_5 + 44_5 + 13_5$ (6) $401_5 + 123_5 + 240_5$

Solve the following and verify the answer by converting the numbers into base-five System.

(7) $31 - 25$ (8) $103 - 34$

Solve the following and then verify the solution by converting the numbers into decimal system.

(9) $(43)_5 - (34)_5$ (10) $(110)_5 - (43)_5$

(11) $(341)_5 - (212)_5$ (12) $(423)_5 - (404)_5$

(13) Add $(432)_5$ to $(103)_5$ and subtract $(434)_5$ from the sum.

(14) The sum of two numbers in Base-five is $(431)_5$. One of the numbers is $(214)_5$. Find the other.

(15) Convert the numbers used in the following expressions into those of base-five and solve

$$21 + 34 - 108 + 62$$

Convert the result so obtained into Decimal system and thus check your answer.

1.5. MULTIPLICATION IN BASE-FIVE SYSTEM

If we have to find the product of $(4)_5$ and $(3)_5$, we multiply 4 by 3 and get 12. As we already know, in base-five system, 12 is written as 2 units and two fives i.e. $12 = (22)_5$

$$\therefore \quad (4)_5 \times (3)_5 = (22)_5$$

Similarly to find the product of $(3)_5$ and $(2)_5$, we multiply 3 by 2 and get 6. In base-five system, 6 is written as $(11)_5$:

$3_5 \times 2_5 = 11$

Multiplication table in base-five system is given below:

x	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11_5	13_5
3	0	3	11_5	14_5	22_5
4	0	4	13_5	22_5	31_5

Example 1

Solve : $341_5 \times 4_5$

Solution

$$\begin{array}{r} 341_5 \\ \times\ 4 \\ \hline 3014_5 \end{array}$$

Explanation:

$4 \times 1 = 4_{10} = 4$

So in the product (4) will take the place of units.

$4 \times 4 = 16_{10} = 31$

(1) Takes the second place on the left and 3 is to be carried to the third place.

$4 \times 3 = 12_{10} = 22_5$

We already have 3 to carry to the third place. So add 3 to 22_5. This gives us 30_5.

Hence the product is 3014_5.

Let us verify the answer

$$341_5 = 3 \times 5^2 + 4 \times 5^1 + 1 \times 5^0$$

$$= 75 + 20 + 1 = 96$$

$$4_5 = 4$$

$$341_5 \times 4_5 = 96 \times 4 = 384$$

We convert this product (384) to Base-five system to see whether it comes to be the product already obtained i.e. 3014

Now

$$\begin{array}{r|l} 5 & 384 \\ \hline 5 & 76-4 \\ \hline 5 & 15-1 \\ \hline & 3-0 \end{array}$$

$\therefore$ 384 = 3014

The verification shows that the answer is correct.

Example 2

Solve :

$324_5 \times 23_5$

Solution :

$$\begin{array}{r} 324_5 \\ 23_5 \\ \hline 2032 \\ 1203 \\ \hline 14112_5 \end{array}$$

EXERCISE NO. 1.4

Find the product of the following in Base-Five system and then verify the answers by changing them into decimal system:

(1) $243_5 \times 2_5$

(2) $431_5 \times 4_5$

(3) $240_5 \times 32_5$

(4) $342_5 \times 43_5$

(5) $443_5 \times 241_5$

1.6. BASIC OPERATIONS IN BASE-TWO SYSTEM

Basic operations in Base-two system can be performed on the pattern of base-five In Base-two system we use two digits namely

0 and 1. Addition and multiplication tables are given below.

Multiplication Table in Base-Two System

x	0	1
0	0	0
1	0	1

Addition Table in Base-two System

+	0	1
0	0	1
1	1	$(10)_2$

Example 1

Find the sum :

$(10101)_2 + (1101)_2$

Solution

$$\begin{array}{r} (10101)_2 \\ +\ (1101)_2 \\ \hline (100010)_5 \end{array}$$

We have used the rule $1 + 1 = (10)_2$

Verification of the answer.

$$(10101)_2 = 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$

$$= 16 + 0 + 4 + 0 + 1 = (21)_{10}$$

$$(1101)_2 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$

$$= 8 + 4 + 0 + 1 = (13)_{10}$$

Thus $(10101)_2 + (1101)_2 = (21)_{10} + (13)_{10} = (34)_{10}$

Converting $(34)_{10}$ to base-two system.

$$\begin{array}{r|l} 2 & 34 \\ \hline 2 & 17 - 0 \\ \hline 2 & 8 - 1. \\ \hline 2 & 4 - 0 \\ \hline 2 & 2 - 0 \\ \hline & 1 - 0 \end{array}$$

$\therefore 34 = (100010)_2$

Hence the result is correct.

Example 2

Subtract $(11010)_2$ from $(100101)_2$

Solution

$$\begin{array}{r} (100101)_2 \\ -(11010)_2 \\ \hline (1011)_2 \end{array}$$

Using the rule that
$10_2 - 1_2 = 1_2$

Verification of the answer.

$$\begin{aligned} (100101)_2 &= 1 \times 2^5 + 0 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 \\ &\quad + 0 \times 2^1 + 1 \times 2^0 \\ &= 32 + 0 + 0 + 4 + 0 + 1 = (37)_{10} \\ (11010)_2 &= 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 \\ &= 16 + 8 + 0 + 2 + 0 = (26)_{10} \end{aligned}$$

So $(100101)_2 - (11010)_2 = (37)_{10} - (26)_{10} = (11)_{10}$

By converting $(11)_{10}$ into Base-two system.

2	11
2	5 – 1
2	2 – 1
	1 – 0

$(11)_{10} = (1011)_2$

Hence the answer is correct.

Example 3

Solve :

$(11101)_2 \times (101)_2$

Solution

$$\begin{array}{r} (11101)_2 \\ (101)_2 \\ \hline 11101 \\ 111010 \\ \hline (10010001)_2 \end{array}$$

Verification of the answer

$$(11101)_2 = 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$

$$= 16 + 8 + 4 + 0 + 1 = (29)_{10}$$

$$(101)_2 = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$

$$= 4 + 0 + 1 = (5)_{10}$$

$$(11101)_2 \times (101)_2 = (29)_{10} \times (5)_{10} = (145)_{10}$$

Now we have to check whether $(145)_{10}$ when converted to base-two gives us $(10010001)_2$

2	145
2	72 – 1
2	36 – 0
2	18 – 0
2	9 – 0
2	4 – 1
2	2 – 0
	1 – 0

$(145)_{10} = (10010001)_2$. So the answer is correct.

EXERCISE No. 1.5

Solve the following in Base-Two, Then convert into Decimal System to check your answer :-

(1) $(101)_2 + (11)_2$

(2) $(110)_2 + (1101)_2$

(3) $(111)_2 + (101)_2 + (10)_2$

(4) $(1110)_2 - (101)_2$

(5) $(1110) - (1101)_2$

(6) $(1101)_2 + (10)_2 - (111)_2$

(7) $(101)_2 \times (11)_2$

(8) $(1101)_2 \times (101)_2$

(9) $(11011)_2 \times (1110)_2$

(10) Add $(1101)_2$ to $(110)_2$ and subtract $(1111)_2$ from the sum.

(11)Solve:-$(11101)_2 \times (110)_2 + (1010)_2 \times (111)_2$

(12)Simplify:-$(1110)_2 \times (101)_2 + (101)_2 \times (10)_2 - (110)_2 \times (1010)_2$

CHAPTER 2

SETS, REAL NUMBERS
AND
CO-ORDINATE SYSTEM

2.1

In the previous classes you have studied about the set and its members. Usually three methods are used to represent the sets.

(i) TABULATION METHOD

Under this method the members of a set are tabulated within the Braces e.g.

N = {1, 2, 3, ... } M = {1, 2, 3, ... , 49 },
L = {a, b, c, } and K = {1/2, 2/4, 3/6, ... }

(ii) DESCRIPTIVE METHOD

In this method the members of the set are described in words, with regard to the common property in them. For example, the above mentioned sets L, M, N, K can be described in words as follows:

L = The set of first three Roman Alphabets.
M = The set of natural members less than 50.
N = The set of all natural numbers.
K = The set of fractions equivalent to 1/2.

The Descriptive Method is useful to represent the sets in which the number of members is fairly larger.

(iii) SET BUILDER NOTATION

M = [1, 2, 3, ... 49]

M can be represented under Set Builder Notation as $M = \{x \mid x < 50$ and x is a natural number$\}$ and is read as "M is the set of all d x such that x is less than 50 and x is a natural number."

Similarly "N = set of natural numbers" can be represented under Set Builder Notation, as

"N = [F.C |ℓ.c is a natural number]"

In this book, for the sake of convenience, specific symbols have been assigned to some of the sets. In future, whenever needed, these specific symbols will be used for the respective sets. These are as follows:

R = The set of all Real Numbers.
N = The set of all Natural Numbers.
Z = The set of all Whole numbers.
I = The set of all Integers.
Q = The set of all Rational Numbers.
P = The set of all Prime Numbers.

2.2 OVERLAPPING SETS AND DISJOINT SETS

1. Two sets are overlapping, if:
   (i) They have at least one element in common, and
   (ii) None of them is the sub-set of the other.

   A = {1, 2, 3, ... , 10} and B = {5, 10, 15, ... ,100} are overlapping sets.

2. "Two sets are called Disjoint if they have no element in Common," e.g.

L = The set of all positive integers and M = [all negative integers] are disjoint sets, because no number can be both positive as well as negative.

2.3 PROPER AND IMPROPER SUB-SETS

We have stuided about the sub-sets in the previous classes. Now we shall learn about proper and improper sub-sets. We have learnt that;

"If each element of a set Y is also on element of another set X, then Y is said to a the sub-set of X"

Suppose $X = \{1, 2, 3, \ldots\ldots, 100\}$

and $Y = \{1, 2, 3, \ldots\ldots, 10\}$, **The Y is a sub-set of X**

It can very easily by seen that both X and Y are sub-sets of N (the set of natural numbers). Moreover, X, Y and N are sub-sets of I (the integers).

In the light of the definition of a sub-set, it is easy to deduce that in fact I is the sub-set of I (i.e. sub-set of itself), because each element of I is an element of I. In general we say that "each [illegible] set is a sub-set of itself."

Sub-sets of this kind are called 'IMPROPER SUB-SETS', e.g., if A = [a, b, c, d] and B = [c, d, a, b], then A is the improper sub-set of B and similarly B is improper sub-set of A. There is only one improper sub-set of a set and the sub-sets other than that are called 'PROPER SUB-SETS'. For example if X = [2, 4, 6]. The its sub-sts are

"{ }, {2}, {4}, {6}, {2, 4}, {2, 6}, {2, 4, 6}

The last one is the improper sub-set of X, while all the rest are proper sub-sets.

" [illegible] or empty set is a sub-set of every set."

Moreover X, Y and N are proper sub-sets of I and I is the improper sub-set of I.

The symbol for 'proper sub-set is '$\subset$'.

The symbol '$\subseteq$' is used for any kind of sub-set. Thus we may write the above statements as :-

$X \subset I$, $Y \subset I$, $N \subset I$, $X \subset N$, $Y \subset N$ and $Y \subset X$.

We shall write $I \subseteq I$ for I is the improper subset of I.

Similarly $N \subseteq N$, $X \subseteq X$, $Y \subseteq Y$.

EXERCISE NO 2.1

Represent the following sets in Set Builder Notation:

(i) $A = \{1, 2 \ldots\ldots 100\}$

(ii) B = The set of the sides of triangle ABC

(iii) $C = \{-1, -2, \ldots\ldots, 1000\}$

(iv) $D = \{2, 4, 6, \ldots\ldots, 5000\}$

(v) E = The set of all even integers.

(vi) O = The set of all odd integers.

(vii) F = The set of all factors of 24.

(viii) Q = The set of all rational numbers.

2.4 FINITE AND INFINITE SETS

If the number of elements of a set is limited, it is said to be a finite set. For example

$K = \{2, 4, 6, 8\}$

$L = \{a, e, i, o, u\}$

$M = \{a, b, c, d, \ldots\ldots\ldots, z\}$

$A = \{1, 2, 3, \ldots\ldots\ldots, 1000\}$

$B = \{x \mid x \in N, 1 \leqslant x \leqslant 100\}$

$C = \{Y \mid \ell.c \in P, 1 < Y < 20\}$

$D = \{a \mid a \in I, -1 \leqslant a \leqslant 10\}$

E = Set of multiples of 16

F = Set of natural numbers less than 500

G = Set of prime numbers less than 50

H = Set of integers less than 25 and greater than 5.

If the number of elements of a set is unlimited, it is called an infinite set. For example

$X = \{5, 10, 15, 20, \ldots\ldots\ldots\}$

$Y = \{0, \pm 10, \pm 20, \pm 30, \ldots\ldots\ldots$

Z = Set of multiples of 6

J = Set of all natural numbers greater than 10.

S = Set of all rational numbers less than ½.

In addition to these sets, I, N, O, P, E, Q, Q′, R are also infinite sets.

2.5 POWER SET

The power set of a set X is the set of all the subsets of X. The power set of X is denoted by P (X).

Example 1. Find the power set of $A = \{2, 3\}$

Solution :- All the subsets of A are:

$\phi, \{2\}, \{3\}, A$

Hence $P(A) = \{\phi, \{2\}, \{3\}, A\}$

Example 2. Find the power set of $S = \{0, 1, 2\}$

Solution :- All the subsets of S are:

$\phi, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, S.$

Hence $P(S) = \{\phi, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, S\}$

Example 3. If $A = \{a\}$, find P(A).

Solution :- All the subsets of A are:

$\phi, \{a.\}$

$\therefore$ $P(A) = \{\phi, \{a\}\}$.

Example 4. Find the power set of $B = \{3, 6, 9, 10\}$

Solution :-

$P(B) = \{\phi, \{3\}, \{6\}, \{9\}, \{10\}, \{3, 9\}, \{3, 10\}, \{6, 9\}, \{6, 10\},$

$\{9, 10\}, \{3, 6, 9\}, \{3, 6, 10\}, \{3, 9, 10\}, \{6, 9, 10\}, B\}.$

It may be noted that:-

(i) If there is one element in a set, its power set will have 2 elements,

(ii) If there are two elements in a set, its power set will have 4 elements,

(iii) If there are three elements in a set, its power set will have 8 elements,

(iv) If there are four elements in a set, its power set will have 16 elements

If $C = \phi$, $P(C) = \{\phi\}$. This shows that the power set of EMPTY SET has one element "ϕ". Therefore, the pwer set of an empty set is not an empty set

EXERCISE NO. 2.2

1. Which of the following sets are finite and which are infinite.

$A = \{0, 1, 2, 3, 4, 5\}$

$B = \{-1, -2, -3, \ldots\ldots\ldots\ldots, -20\}$

$C = \{-1, -2, -3, \ldots\ldots\ldots\ldots\ldots\ldots\}$

$D = \{0, 2, 4, 6, 8, \ldots\ldots\ldots\ldots\ldots\ldots\}$

$E = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots\ldots\ldots\ldots\ldots\ldots\}$

F = Set of multiples of 20

G = Set of divisors of 20

H = Set of rational numbers greater than 50.

J = Set of real numbers less than zero.

K = Set of natural numbers less than 100.

L = Set of integers less than 100.

$M = \{x \mid x \in Q, 0 \leq x \leq 1\}$

$N = \{y \mid s \in R, -1 \leq s \leq 1\}$

$S = \{s \mid s \in N, 1 \leq s \leq 2\}$

$T = \{t \mid t \in N, 100 \leq t \leq 200\}$

$V = \{v \mid v \in Z, 0 \leq v \leq 1000\}$

2. Find power sets of the following sets:-

$A = \{2\}$, $B = \{1\}$, $C = \{0\}$, $D = \phi$,

$E = \{1, 2\}$, $F = \{0, 1\}$ $G = \{-1, 1\}$,

$H = \{\frac{1}{2}, \frac{1}{2}\}$, $G_1 = \{\frac{1}{3}, \frac{2}{5}\}$, $H_1 = \{-\frac{1}{5}\}$

$J = \{0, 1, 2\}$, $K = \{2, 4, 6\}$, $L = \{-1, -2\}$

$M = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\}$, $N = \{\sqrt{2}, \sqrt{3}, \sqrt{5}\}$

$S = \{2, \frac{1}{2}, \sqrt{2}\}$, $T = \{1, 2, 3, 4\}$

$V = \{0, 1, -1, -2\}$ $W = \{\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\}$

$X = \{\frac{1}{2}, \sqrt{2}, 0, 2\}$

2.6 SETS OF NUMBERS

As you remember, we started the study of Mathematics with counting numbers 1, 2, 3, 4, 5 . . . . . . we called them 'Natural Numbers': The set of all natural numbers is denoted by 'N'

i.e. $N = \{1, 2, 3, 4, \ldots\ldots\}$

We included 0 in this set of numbers and named it as the set of WHOLE NUMBERS which is denoted by Z.

i.e. $Z = \{0, 1, 2, 3, 4, 5, \ldots\ldots\}$

We extended the number system still further to include negative numbers and called it the set of Integers. We denoted it by I

i.e. $I = \{\ldots\ldots -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\ldots\}$

To broaden the horizon further, we incorporated fractions in the family of numbers and called the resulting set by the name of set of Rational Numbers and denoted it by Q

i.e $Q = \{\frac{p}{q} \mid p, q \in I, q \neq 0\}$

2.7 IRRATIONAL NUMBERS

There are many numbers which cannot be written in the form p/q where p and q are integers and $q \neq 0$. For instance, the square root of a number which is not a perfect square and the cube root of a number that is not a perfect cube, are numbers which cannot be written in the form $\frac{p}{q}$ when p, q, Q, I and $q \neq 0$

$\sqrt{2}, \sqrt{5}, \sqrt{7}, \sqrt[3]{11}, \sqrt[4]{15}, \sqrt[5]{17}$ cannot be expressed in the form $\frac{p}{q}$, where $p, q \in I$ and $q \neq 0$.

Numbers which cannot be written in the form $\frac{p}{q}$ (where p and q are integers and $q \neq 0$) are called irrational numbers. The set of irrational numbers in denoted by Q′.

2.8 REAL NUMBERS

The set of Real Numbers R is the union of the set of Rational numbers Q and the set of Irrational numbers Q′.
i.e., $R = Q \cup Q'$.

Hence we can say that

$$N \subset Z \subset I \subset Q \subset R \text{ and also } Q' \subset R.$$

Now we shall study this system of numbers in detail.

2.9 PROPERTIES OF THE REAL NUMBERS

We shall study here some of the properties of the set R which are needed for our present study of Mathematics.

1. Closure Property w.r.t. Addition

(w.r.t. means with respect to).

The sum of any two real numbers is a real number,

e.g. $1 + \frac{2}{3} = \frac{5}{3}, \frac{4}{5} + \frac{1}{2} = \frac{13}{10}$,

$3 + 1 = 4, 1 + 1.73 = 2.73$ etc.

In symbols we express the property as

$\forall\ a, b \in R,\quad a + b \in R$ " $\forall$ " means 'for all of'

It is read as "For all of a and b belonging to R, the sum of a and b i.e. a + b also belongs to R".

2. **Commutative Property w.r.t. Addition**

If a and b are any two real numbers, then $a + b = b + a$.

If other words 'the sum of any two real numbers a and b added in any order is the same.'

Symbolically we express this property as

$$\forall a, b \in R, \quad a + b = b + a.$$

3. **Associative Property w.r.t. Addition**

If a, b, c are any real numbers.

then $(a + b) + c = a + (b + c)$.

In other words 'either we add c to the sum of a and b, or we add a to the sum of b and c, the resultant sum in both the cases is the same.

Symbolically we write

$$\forall a, b, c \in R, \quad (a + b) + c = a + (b + c).$$

4. **Existence of Additive Identity**

There is a real number '0' such that for any real number 'a'

$$a + 0 = 0 + a = a$$

0 is called the additive identity.

Symbolically $\forall a \in R \ \exists \ 0 \in R$ s. t.

$$a + 0 = 0 + a = a$$

'$\exists$' stands for 'there exists'

5. **Existence of Additive Inverse**

For every real number 'a' there exists a real number '$-a$' (negative a) such that

$$a + (-a) = (-a) + a = 0$$

a + (– a) can also be written as a–a 'a' and '–a' are the additive inverses of each other.

6. **Closure Property w.r.t. Multiplication**

The product of any two real numbers is a real number.

Symbolically

$\forall_{a,b} \in R$, a x b is a real number 'a x b' is also written as 'ab'.

7. **Commutative Property w.r.t. Multiplication**

For any two real numbers a and b ab = ba.

Symbolically $a, b \in R$, ab = ba

8. **Associative Property w.r.t. Multiplication**

For any real numbers a, b, c

(ab) c = a (bc).

Symbolically

$\forall_{a,b,c} \in R$, (ab) c = a (bc).

9. **Existence of Multiplicative Identity**

There exists a real number '1' such that for any real number 'a'

a x 1 = 1 x a = a.

Symbolically

$\forall_a \in R \; \exists \; 1 \in R$ s.t. 1 x a = a x 1 =

1 is called the multiplicative identity.

10. **Existence of Multiplicative Inverse**

For every non-zero real number 'a' there exists a number 1/a such that

$$a \times \frac{1}{a} = \frac{1}{a} \times a = 1$$

'a' and '1/a' are the multiplicative inverse of each other.

Remember that multiplicative inverse exists for all real numbers except 'O'. Though 'O' is a real number but its inverse does not exist.

In other words, there exists no number which when multiplied by zero gives '1' as product.

11. Distributive Property of Multiplication w.r.t. Addition

If a, b, c are any real numbers, then

$$a(b + c) = ab + ac \text{ and } (b + c)\,a = ba + ca.$$

Symbolically

$$\forall a, b, c \in R,\ a\,(b + c) = ab + ac$$
$$(b + c)\,a = ba + ca$$

12. Trichotomy Property

For any real number ' a ' one and only one of the following three relations will be true.

either $a > 0$ or $a = 0$ or $a < 0$.

2.10 PROPERTIES OF EQUALITY OF REAL NUMBERS

1. Reflexive Property

For every real number x

$$x = x$$

2 Symmetric Property.

If w and f are any two real numbers and

$$x = y$$

then $y = x$

Symbolically

$$x, y \in R \quad x = y \Rightarrow y = x$$

The symbol '⇒' stands for 'implies that'. This also stands for "if . . . . then . . . ." e.g.

a = b ⇒ b = a (i.e. if a = b, then b = a)

3. **Transitive Property:-**

If x, y and z are any three real numbers such that

$$x = y \text{ and } y = z$$

then $x = y$

Symbolically

$$\forall x, y, z \in \text{R} \quad x = y \wedge y = z \Rightarrow x = z$$

Note:- Symbol '$\wedge$' stands for 'and'

4. **Additive Property**

If x, y, z are any three real numbers, and if $x = y$

then $x + z = y + z$

Symbolically

$$\forall x, y, z \in \text{R} \quad x = y \Rightarrow x + z = y + z$$

5. **Multiplicative Property**

If x, y, z are any three real numbers then

$$x = y \Rightarrow xz = yz \text{ and } zx = zy.$$

6. **Cancellation Law w.r.t. Addition**

If x, y, z are any three real numbers, then

$$x + y = z + y \Rightarrow x = y \text{ and } y + x = z + y \Rightarrow x = y$$

7. **Cancellation Law w.r.t. Multiplicative**

If x, y, z are any three real numbers and $z \neq 0$, then

$$xz = yz \Rightarrow x = y$$

and $zx = zy \Rightarrow x = y$

The following results can be deduced with the help of the above properties:

(i) $\forall x \in \text{R} \quad x \times 0 = 0 \times x = 0$

(ii) $\forall x, y \in \text{R} \quad xy = 0 \Rightarrow x = 0$ or $y = 0$

(iii) $\forall x \in R,\ -(-x) = x.$

(iv) $\forall x \in R,\ \dfrac{1}{\dfrac{1}{x}} = x$

(v) $\forall x, y \in R,\ (-x)\,y = x\,(-y) = -xy$

(vi) $\forall x, y \in R,\ (-x)\,(-y) = xy$

(vii) $\forall x, y, z, s \in R,\ \dfrac{x}{y} = \dfrac{z}{s} \Rightarrow xs = yz \qquad \begin{matrix} y \neq 0 \\ s \neq 0 \end{matrix}$

(viii) $\forall x, y, z, s \in R,$ and $y, z, s \neq 0\ \dfrac{xz}{yz} = \dfrac{x}{z} \times \dfrac{z}{s}$

(ix) $\forall x, y, x \in R,\ \dfrac{x}{y} = \dfrac{kx}{ky},$ where $y \neq 0,\ k = 0.$

(x) $\forall x, y, z, s \in R,\ \dfrac{\dfrac{x}{y}}{\dfrac{z}{s}} = \dfrac{xs}{yz}$, where $y \neq 0,\ z \neq 0,$

(xi) $x, y, z \in R,\ \dfrac{x}{z} + \dfrac{y}{z} = \dfrac{x+y}{z}$, where $z \neq 0.$

2.11 PROPERTIES OF INEQUALITY OF REAL NUMBERS

1. Transitive Property

If a, b, c are any three real numbers, then

(i) $a > b$ and $b > c \Rightarrow a > c.$

(ii) $a < b,\ b < c \Rightarrow a < c.$

2. Additive Property

If a, b, c are any three real numbers, then

(i) $a > b \Rightarrow a + c > b + c$ and $c + a > c + b.$

(ii) $a < b \Rightarrow a + c < b + c$ and $c + a < c + b.$

3. Multiplicative Property

(a) If a, b, c are any three real numbers and $c > 0$ then

 (i) $a > b \Rightarrow ac > bc$ and $ca > cb.$

 (ii) $a < b \Rightarrow ac < bc$ and $ca < cb.$

(b) If a, b, c are any three real numbers and $c < 0$ (i.e. c is a negative number), then

(i) $a > b \Rightarrow ac < bc$ and $ca < cb$.

(ii) $a < b \Rightarrow ac > bc$ and $ca > cb$.

(Remember that all +ve integers are greater than zero and all −ve integers are less than zeros).

Note: Though it seems too difficult to remember all the properties but if you concentrate a little you will come to know that many of the properties are already known to you. You have just been made conscious of these properties.

The properties of equality of real numbers with regard to subtraction and division can very easily be deduced because subtraction and division are the inverse operations of addition and multiplication respectively. The teachers should help the students in deducing these properties.

EXERCISE NO. 2.3

1. Are the following sets closed (i.e. have closure property) with regard to multiplication ?

(i) $\{0, 1\}$, (ii) $\{1, -1\}$, (iii) $\{0\}$, (iv) $\{1\}$,

(v) $\{1, -\sqrt{-1}, \sqrt{-1}\}$.

2. Which of the sets given in Question 1 are closed with respect to addition ?

3. Are the following sets closed (i.e. have closure property) with respect to:

(a) Addition (b) Multiplication (c) Subtraction

(d) Division.

(i) Q, (ii) Z, (iii) N, (iv) P, (v) I, (vi) The set of even integers, (vii) The set of odd integers.

4. Check yourself and find which of the properties of R do the sets Q, I, Z and N possess ?

5. Name the properties used in each of the following equalities.

(i) $3 + 7 = 7 + 3.$

(ii) $\frac{1}{2} + \frac{2}{3} = \frac{2}{3} + \frac{1}{2}.$

(iii) $(2 + 4) + 1 = 2 + (4 + 1).$

(iv) $\left(\frac{2}{5} + \frac{3}{7}\right) + \frac{1}{4} = \frac{2}{5} + \left(\frac{3}{7} + \frac{1}{4}\right)$

(v) $5 + 0 = 0 + 5 = 5.$

(vi) $-2 + 0 = 0 + (-2) = -2.$

(vii) $\left(-\frac{1}{3}\right) + \frac{1}{3} = \frac{1}{3} + \left(-\frac{1}{3}\right) = 0.$

(viii) $\sqrt{2} + (-\sqrt{2}) = 0,$

(ix) $4 \times 5 = 5 \times 4.$

(x) $\frac{1}{3} \times \frac{2}{5} = \frac{2}{5} \times \frac{1}{3}$

(xi) $(3 \times 7) \times 9 = 3 \times (7 \times 9)$

(xii) $\left(\frac{2}{7} \times \frac{3}{7}\right) \times 6 = \frac{2}{7} \times \left(\frac{3}{7} \times 6\right)$

(xiii) $3 \times 1 = 1 \times 3 = 3.$

(xiv) $-\sqrt{2} \times 1 = 1 \times -\sqrt{2} = -\sqrt{2}.$

(xv) $5 \times \frac{1}{5} = \frac{1}{5} \times 5 = 1$

(xvi) $(-\sqrt{3})\left(-\frac{1}{\sqrt{3}}\right) = 1$

(xvii) $3(4 + 5) = 3 \times 4 + 3 \times 5.$

(xviii) $\frac{1}{2}\left(\frac{2}{3} + \frac{5}{7}\right) = \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{5}{7}$

6. Name the properties used in each of the following inequalities'

(i) $4 > 3, 7 > 2 \Rightarrow 11 > 5$.

(ii) $\frac{1}{3} < \frac{1}{2}, \frac{2}{3} < \frac{3}{4} \Rightarrow 1 < \frac{5}{4}$

(iii) $9 > 6 \Rightarrow 12 > 9$.

(iv) $\frac{3}{4} > 2 \Rightarrow 1 < 2\frac{1}{4}$

(v) $6 > 5 \Rightarrow 24 > 20$.

(vi) $\frac{1}{2} < \frac{2}{3} \Rightarrow 3 < 4$.

(vii) $-3 < 5 \Rightarrow 9 > -15$.

(viii) $\frac{3}{4} > \frac{1}{5} \Rightarrow \frac{-21}{4} < \frac{-7}{5}$

7. Find out the fallacy in the solution of the following problem:

Let $x = y$ Now $x = y \Rightarrow x \cdot x = x \cdot y$

$$\Rightarrow x^2 = xy$$

$$\therefore \quad x^2 - y^2 = xy - y^2 \quad (\because \ x^2 = xy)$$

$$\Rightarrow (x - y)(x + y) = y(x - y)$$

$$\Rightarrow x + y = y \Rightarrow x + x = x \quad \because \ x = y$$

$$\Rightarrow 2x = x$$

$$\Rightarrow 2 = 1 \ ?$$

2.12 ORDERED PAIRS

A pair of elements a, b, written (a, b) is called an ordered pair. It is called an ordered pair because the order of the elements is important, that is to say that an ordered pair (a, b) is different from an ordered pair (b, a). In the first case the first element is 'a' and the second is 'b' while in the second case the first element is 'b' and the second is 'a'.

In fact (a, b) = (b, a) ⇔ a = b

['$\Longrightarrow$' Stands for 'double implication', that is to say if (a, b) = (b, a) then a = b and on the other hand, if a = b then (a, b) = (b, a). '$\Leftrightarrow$' is also read as ' if and only if '] . The elements of the ordered pair (a , b) are usually real numbers but it is not always true because the elements of an ordered pair can be taken other than the real numbers. However in this book we shall only be concerned with the real numbers to be the elements of ordered pairs.

AXIOM: Two ordered pairs (a, b) and (c, d), will be equal 'if and only if' (can also be written as iff) a = c and b = d.

Symbolically $(a, b) = (c, d) \Longleftrightarrow a = c$ and $b = d$.

2.13 THE CARTESIAN PRODUCT

The cartesian product of two sets A and B is the set of all the ordered pairs (a, b) such that $a \in A$ and $b \in B$. The Cartesian product of A and B is written $A \times B$.

Symbolically $A \times B = \{(a, b \mid a \in A \text{ and } b \in B)\}$

It is important to note that though the symbol 'x' is used to denote multiplication but in A x B this symbol does not mean multiplication in that sense.

We can also find the Cartesian product of any set S with itself. In this way

$S \times S = \{ (x, y) \mid x \in S \text{ and } y \in S \}$

Example 1

Find $A \times B$ if $A = \{0, 1\}$, $B = \{1, 2, 3\}$

Solution

We know that

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

$$\therefore \quad A \times B = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3)\}$$

In other words, A x B consists of those ordered pairs whose first elements belong to A and the second elements belong to B.

$$B \times A = \{(1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1)\}$$

We see that

$$A \times B \neq B \times A$$

Example 2

If $S = \{ \frac{1}{2}, -1, 0 \}$, find $S \times S$

Solution

$$S \times S = \{ (x, y) | x \in S| , y \in s \}$$

$$S \times S = \{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2} - 1), (\frac{1}{2}\ 0), (-1, \frac{1}{2}), (-1, -1),$$

$$(-1, 0), (0, \frac{1}{2}), (0, -1), (0, 0)\}$$

2.14 CO-ORDINATE SYSTEM

Co-ordinate system is such a device which helps to determine the position of an object. Some examples are given below:

1. A residence is usually located with the help of house number and street number.

2. A student is identified with the help of class number and roll number.

3. The temperature is given in degrees above or below zero.

4. The position of a ship is located with the help of latitudes and longitudes.

Without going into details we conclude that in order to represent points on a line we need only one ordered pair or one lable as we have been representing numbers on lines in the previous classes.

2.15 CARTESIAN CO-ORDINATE SYSTEM FOR PLANE

The system of using ordered pairs of real numbers for representing the position of a point in a plane was introduced by a French Mathematician, Descartes and is, therefore, known as Cartesian System.

In this system we take two lines, one of which is vertical and the other is horizontal. The point where the two lines meet is called origin. The origin is denoted by "O". The horizontal line is called x-axis and the vertical line is called y-axis. The x-axis is also denoted by XX′ and y-axis by YY′.

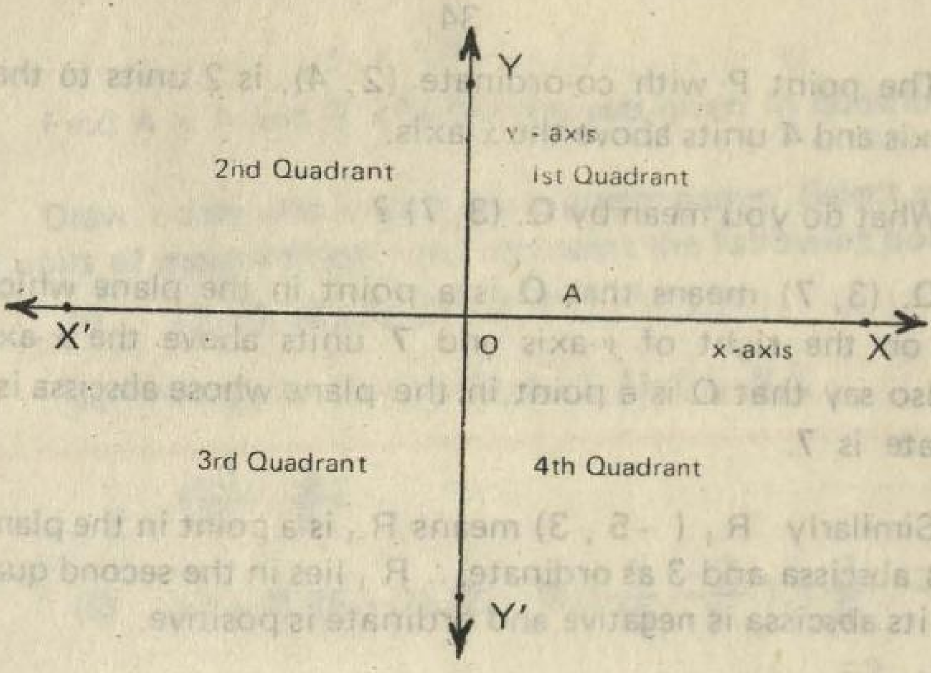
Select any point A on the x-axix and on the right of 'O' (the origin), take m $\overline{OA}$ as the unit of the length. With this unit of length measure the distances along x-axis and y-axix. In this way an orderd pair (a, b) is associated with one point in the plane; 'a' is the x-co-ordinate and 'b' is the y-co-ordinate of the point. The first and the second elements of an ordered pair associated with a point always represent the distances of the point along x-axix and y-axis respectively.

The elements of an ordered pair (a , b) are called the co-ordinates of the point associated with it. ' a ' is the x-co-ordinate or abscissa and 'b' is the y-co-ordinate or ordinate . It is a convention that the distances on the right of y-axis are taken positive and those on the left of y-axis are taken negative. Similarly the distances above the x-axis along OY are taken positive and those below the x-axis are taken negative.

The two axes divide the plane into four parts called quadrants.

XOY is the first quadrant, X′OY is the second quadrant X′OY′ is the third quadrant and XOY′ is the fourth quadrant.

It is now evident that in the first quadrant abscissa is positive and ordinate is positive.

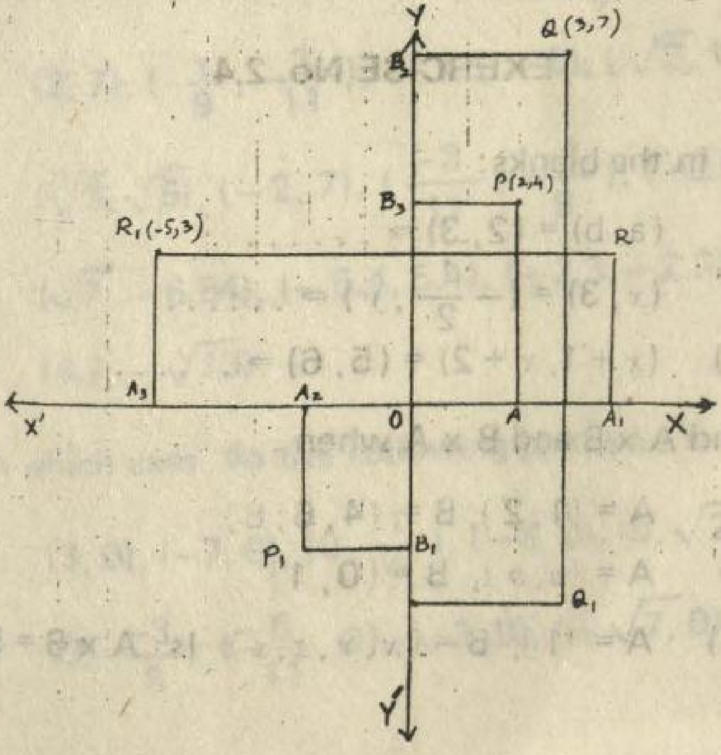

Y
y - axis
2nd Quadrant
Ist Quadrant
A
X'
O
x-axis
X
3rd Quadrant
4th Quadrant
Y'

In the second quadrant abscissa is negative and ordinate is positive.

In the third quadrant abscissa is negative and ordinate is negative.

In the fourth quadrant abscissa is positive and ordinate is negative.

The points in a plane are denoted by A, B, C . . . . X, Y, Z, as we have been doing in the case of a line. The above concepts are explained with the help of the following figure.


Y
Q(3,7)
B
P(2,4)
B₃
R₁(-5,3)
R
A₃
A₂
O
A
A₁
X'
X
P₁
B₁
Q₁
Y'

The point P with co-ordinate (2, 4), is 2 units to the right of y-axis and 4 units above the x-axis.

What do you mean by Q. (3, 7) ?

Q. (3, 7) means that Q is a point in the plane which is 3 units on the right of y-axis and 7 units above the x-axis. We can also say that Q is a point in the plane whose abscissa is 3 and ordinate is 7.

Similarly R_1 (−5 , 3) means R_1 is a point in the plane with −5 as abscissa and 3 as ordinate. ∴ R_1 lies in the second quadrant. Thus its abscissa is negative and ordinate is positive.

Write the co-ordinate of P_1, Q_1, R yourself.

What are the co-ordinates of A ?

We see that abscissa of A is 2, but what is its ordinate? or what is the distance of A from x-axis. Obviously, since the point A lies on the x-axis, its distance from x-axis is zero. Hence the ordinate of the point A is 'zero' and the co-ordinates of A are (2, 0).

Now you can easily find the co-ordinates of the points A_1, A_2, A_3, B_1, B_2, B_3. If not, ask your teacher.

EXERCISE No. 2.4

1. Fill in the blanks:

   (i) $(a, b) = (2, 3) \Rightarrow \ldots\ldots$

   (ii) $(x, 3) = (-\frac{1}{2}, y) \Rightarrow \ldots\ldots$

   (iii) $(x+1, x+2) = (5, 6) \Rightarrow \ldots\ldots$

2. Find A x B and B x A when

   (i) $A = \{1, 2\}$, $B = \{4, 6, 8\}$

   (ii) $A = \{a, b\}$, $B = \{0, 1\}$

   (iii) $A = \{1\}$, $B = \{x, y, z, s\}$. Is $A \times B = B \times A$?

3. Find A x A and B x B for the sets given in question 2.

4. Draw x-axis and y-axis on a graph paper. Select some suitable units of measurement and represent the following points:

(i) $(2, 6), (4, 3), (1, 5), (1/10, 2/10), (2/5, 1/5)$.

(ii) $(-2, 7), (-4, -3), (-5, 1), (1/2, 4/5)$,

$(\frac{-3}{5}, \frac{3}{5})$.

(iii) $(3, -2), (4, -1), (2, -6), (\frac{1}{10}, \frac{-2}{5}), (\frac{1}{5}, \frac{-1}{5})$

(iv) $(-2, -5), (-2, -6), (-1, -7), (\frac{-1}{10}, \frac{-2}{10})$,

$(\frac{-4}{5}, \frac{-4}{5})$.

(v) $(3, 0), (-5, 0), (\frac{3}{5}, 0), (\frac{-6}{10}, 0), (1.5, 0)$.

(vi) $(0, 4), (0, -7), (0, 2.3), (0, 1.9), (0, \frac{-7}{10})$.

(vii) $(0, 0)$

5. In which the quadrants do the following points life ?

$(2, 7), (\frac{1}{9}, \frac{3}{11}), (-1.8, -5), (\sqrt{2}, 1)$,

$(\sqrt{3}, \sqrt{5}), (-2, 7), (\frac{-3}{11}, \frac{-1}{9}), (-2.7, 4.8)$,

$(\sqrt{7}, -6.54), (-5.4, 5.4), (-2.3, -2.3)$,

$(4.7, -\sqrt{13})$.

6. On which axes do the following points lie ?

$(3, 0), (-7, 0), (0, \frac{1}{4}), (-8, 0), (0, \sqrt{2}), (0, -\sqrt{5})$,

$(0, \frac{-3}{8}), (\frac{-6}{11}, 0), (\sqrt{3}, 0), (-\sqrt{7}, 0)$.

7. What is the ordinate of the points lying on the x-axis?

8. What is the abscissa of the points lying on the y-axis?

CHAPTER 3

LAWS OF POWERS AND LOGARITHM

We write 3×3 as 3^2. Here 3 is the base and 2 is the exponent. Since $3 \times 3 = 9$, therefore, 9 is second power of 3. We shall learn the laws of powers in this chapter. With their help the operations of multiplication and division become easier. But sometimes the process of simplification of expression becomes so difficult that the laws of exponents do not help much. In such a situation we use the laws of 'logarithm'. This saves time and labour. We shall study the laws of powers first and then those of logarithm.

3.1 LAWS OF POWERS

1. FIRST LAW: (Laws of Product of Powers) Consider the following examples:

$$2^3 \times 2^2 = (2 \times 2 \times 2) \times (2 \times 2)$$

$$= 2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 2^{3+2}$$

$$(-3)^3\ (-3)^4 = [(-3)(-3)(-3)] \times [(-3)(-3)(-3)(-3)]$$

$$= (-3) \times (-3) \times (-3) \times (-3) \times (-3) \times (-3) \times (-3)$$

$$= (-3)^7 = (-3)^{3+4}$$

$$(\frac{1}{5})^2 \times (\frac{1}{5})^4$$

$$= (\frac{1}{5} \times \frac{1}{5}) \times (\frac{1}{5} \times \frac{1}{5} \times \frac{1}{5} \times \frac{1}{5})$$

$$= \frac{1}{5} \times \frac{1}{5} \times \frac{1}{5} \times \frac{1}{5} \times \frac{1}{5} \times \frac{1}{5}$$

$$= (\frac{1}{5})^6 = (\frac{1}{5})^{2+4}$$

We get the following law with the help of these examples:

$$a^r a^s = a^{r+s}$$

where a, r and s are any real numbers, not simultaneously zero.

Similarly $a^r a^s . a^t = a^{r+s+t}$

II. SECOND LAW (Law of Power of Power). Consider the following examples:

$$(3^4)^3 = 3^4 \times 3^4 \times 3^4 = 3^{4+4+4} = 3^{4\times3}$$

$$[(-\frac{1}{2})^3]^5 = (-\frac{1}{2})^3 \times (\frac{1}{2})^3 \times (\frac{1}{2})^3 \times (\frac{1}{2})^3 \times (\frac{1}{2})^3$$

$$= (-\frac{1}{2})^{3+3+3+3+3} = (\frac{1}{2})^{3\times5}$$

$$[(-4)^2]^5 = (-4^2 \times (-4)^2 \times (-4)^2 \times (-4)^2 \times (-4)^2$$

$$= (-4)^{2+2+2+2+2} = (-4)^{2\times5} .$$

So we get the following general law:

$$(a^r)^s = a^{rs}$$

where a, r and s are real numbers, not simultaneously zero.

Some more examples are given below:

$$[(4)^5]^4 = (4)^{5\times4} = (4)^{20}$$

$$[(\frac{1}{5})^2]^3 = (\frac{1}{5})^{2\times3} = (\frac{1}{5})^6$$

$$[(\frac{-1}{2})^3]^3 = (\frac{-1}{2})^{3\times3} = (\frac{-1}{2})^9$$

III. THIRD LAW (Law of the Power of a Product). Consider the following examples:

$$(2 \times 3)^2 = (2 \times 3)\ (2 \times 3) = 2 \times 2 \times 3 \times 3$$
$$= 2^2 \times 3^2$$

$$(\tfrac{1}{2} \times \tfrac{1}{3})^3 = (\tfrac{1}{2} \times \tfrac{1}{3}) \times (\tfrac{1}{2} \times \tfrac{1}{3}) \times (\tfrac{1}{2} \times \tfrac{1}{3})$$

$$= \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{3} \times \tfrac{1}{3} \times \tfrac{1}{3} = (\tfrac{1}{2})^3 \times (\tfrac{1}{3})^3$$

$$[(-3)\ (-2)]^4 = (-3)(-2) \times (-3)(-2) \times (-3)(-2) \times (-3)(-2)$$
$$= (-3) \times (-3) \times (-3) \times (-3) \times (-2) \times (-2) \times (-2) \times (-2)$$
$$= (-3)^4 \times (-2)^4$$

With the help of the above examples we get the following law:

$$(ab)^r = a^r\, .\, b^r$$

where a, b and r are real numbers.

Some more examples are given below:

$$(3 \times 5)^5 = 3^5 \times 5^5$$

$$(-2a)^5 = (-2)^5 \times (a)^5 = (-2)^5 \times a^5$$

We can further extend this law as:

$$(a \times b \times c)^r = a^r \times b^r \times c^r$$

Thus, $(3 \times 4 \times 5 \times 6)^2 = 3^2 \times 4^2 \times 5^2 \times 6^2$

EXERCISE No. 3.1

Simplify:

(1) $2^5 . 2^2$ (2) $3^3 . 3^4 . 3^5$

(3) $x^5 . x^7$ (4) $(\frac{1}{2})^2\ (\frac{1}{2})^3\ (\frac{1}{2})^4$

(5) $bc^2\ . b^2c^3$ (6) $x^2 . x^6 . x^8$

(7) $(2^2)^3$ (8) $(a^4)^3$

(9) $[(-3)^2]^4$ (10) $[(\frac{-2}{7})^2]^9$

(11) $(2 \times 3)^3$ (12) $(\frac{1}{3} \times \frac{1}{5})^2$

(13) $[3 \times (-4)]^2$ (14) $(a^2 b^3)^4$

IV. FOURTH LAW (Law of Power of a Fraction): Consider the following examples:

$$(\frac{2}{3})^3 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{2 \times 2 \times 2}{3 \times 3 \times 3} = \frac{2^3}{3^3}$$

$$(\frac{-5}{7})^4 = (\frac{-5}{7})(\frac{-5}{7})(\frac{-5}{7})(\frac{-5}{7})$$

$$= \frac{(-5)(-5)(-5)(-5)}{7 \times 7 \times 7 \times 7} = \frac{(-5)^4}{7^4}$$

So we get the following law

$$(\frac{a}{b})^r = \frac{a^r}{b^r}, (b \neq o)$$

where a, b and r are real numbers and b is non-zero.

V. FIFTH LAW (Law of division of Powers with the same base). Consider the following examples:

$$\frac{4^5}{4^3} = \frac{4 \times 4 \times 4 \times 4 \times 4}{4 \times 4 \times 4} = 4 \times 4 = 4^2 = 4^{5-3}$$

$$\frac{a^6}{a^2} = \frac{a \times a \times a \times a \times a \times a}{a \times a} = a \times a \times a \times a = a^4 = a^{6-2}$$

So we get the following law:

$$\frac{a^r}{a^s} = a^{r-s} \quad (r > s), (a \neq o) \ldots\ldots \text{V(A)}$$

where a is a non-zero real number and r and s are natural numbers.

Now consider the following two examples:

$$\frac{7^4}{7^4} = \frac{7 \times 7 \times 7 \times 7}{7 \times 7 \times 7 \times 7} = 1$$

$$\frac{a^5}{a^5} = \frac{a \times a \times a \times a \times a}{a \times a \times a \times a \times a} = 1$$

We can say that

$$\frac{a^r}{a^r} = 1 \qquad (a \neq 0) \qquad \text{V(B)}$$

Now consider the following example:

$$\frac{5^3}{5^5} = \frac{5 \times 5 \times 5}{5 \times 5 \times 5 \times 5 \times 5} = \frac{1}{5^2} = \frac{1}{5^{5-3}}$$

$$\frac{a^3}{a^7} = \frac{a \times a \times a}{a \times a \times a \times a \times a \times a \times a} = \frac{1}{a \times a \times a \times a}$$

$$= \frac{1}{a^4} = \frac{1}{a^{7-3}}$$

With the help of these examples we get:

$$\frac{a^r}{a^s} = \frac{1}{a^{s-r}} \quad (r < s)\text{, } (a \neq o \qquad \text{V(C)}$$

V(A), V(B), and V(C) seem to be different results but we can convert them into a simple result if we take the following results to be true which relate to the powers of numbers with exponents as zero or negative numbers.

(i) $a^0 = 1$ $(a \neq 0)$ and a is a real number.

(ii) $a^{-n} = \frac{1}{a^n}$ where n is a natural number.

According to (i) 3°, $(¼)^\circ$, $(\frac{\sqrt{2}}{3})^\circ$, x° $(x \neq 0)$ etc. are all equal to 1.

According to (ii)

$$a^{-4} = \frac{1}{a^4},\ 5^{-3} = \frac{1}{5^3} \text{ and } 2^5 \times 3^{-4} = \frac{2^5}{3^4}$$

In the light of the above results we can write V(A), V(B), V(C) in the form of a single result as

$\frac{a^r}{a^s} = a^{r-s} (a \neq 0)$ where a, r, s are real numbers.

EXERCISE No. 3.2

Simplify:

(1) $(\frac{3}{4})^3$ (2) $(\frac{x}{y})^4$

(2) $(\frac{-p}{q})^3$ (4) $(\frac{2a}{3b})^5$

(5) $(\frac{a^2b^3}{c^4})^2$ (6) $\frac{3^7}{3}$

(7) $\frac{a^3}{a^2}$ (8) $\frac{a^9b^7}{a^3b}$

(9) $\frac{(a+b)^7}{(a+b)^3}$ (10) $\frac{(p+q)^{15}(r+s)^9}{(p+q)(r+s)^2}$

Simplfy or write the following expressions in such a way that the exponents are positive:

(11) $\frac{p^4}{p^7}$ (12) $\frac{b^2 \cdot b^5}{b^8}$ (13) $4a^3b^{-7}$

(14) $\frac{2a^{-3}}{b^{-4}}$ (15) $\frac{8p^{-6}}{-4q^2}$

3.2 SQUARE ROOT

Take the equation $x^2 = 36$. Only 6 and -6 are the two numbers which, when squared give 36. Therefore, the solution set of the given equation is $\{6, -6\}$. Each one of 6 and -6 is called square root of 36. Positive square root is called 'Principal square root' and is written as $\sqrt{36}$. Negative square root is denoted as $-\sqrt{36}$.

So if x is any number such that $x \geq 0$, then $\sqrt{x}$ will always mean principal square root of x, "$\sqrt{\ }$" is radical sign. In $\sqrt{36} = 6$, 6 is principal square root and 36 is "Radicand".

We can also say that $\sqrt{x}$ means a + ve number y which, when squared, is equal to x.

i.e. if $\sqrt{x} = y$, then $x = y^2$

Or we can also say that $\sqrt{x} \times \sqrt{x} = x$

Similarly $\sqrt{2} \times \sqrt{2} = 2$

Below are given some results without proof which help in simplifying the 'radical expression':

(i) If $\sqrt{a} = x$ then $a = x^2$

(ii) $\sqrt{a} \times \sqrt{a} = a$

(iii) $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

(iv) $\dfrac{\sqrt{a}}{\sqrt{a}} = 1$

(v) $\dfrac{a}{\sqrt{a}} = \dfrac{\sqrt{a} \times \sqrt{a}}{\sqrt{a}} = \sqrt{a}$

(vi) $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$

Solved examples:

1. $\sqrt{4} \times \sqrt{9} = 2 \times 3 = 6$
2. $\sqrt{16 \times 25} = \sqrt{16} \times \sqrt{25} = 4 \times 5 = 20$

3. $\sqrt{20} = \sqrt{5 \times 4} = \sqrt{5} \times \sqrt{4} = \sqrt{5} \times 2 = 2\sqrt{5}$

4. $\sqrt{2} \times \sqrt{3} \times \sqrt{7} = \sqrt{2 \times 3 \times 7} = \sqrt{42}$

5. $\dfrac{5}{\sqrt{5}} = \dfrac{\sqrt{5} \times \sqrt{5}}{\sqrt{5}} = \sqrt{5}$

or $\dfrac{5}{\sqrt{5}} = \dfrac{5 \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \dfrac{5\sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \dfrac{5\sqrt{5}}{5} = \sqrt{5}$

6. $3\sqrt{5} + 8\sqrt{5} = (3 + 8)\sqrt{5} = 11\sqrt{5}$

3.3 qth ROOD OF a

We know that there are two square roots ($\sqrt{a}$ and $-\sqrt{a}$) of a number 'a' because square of each one of them equals a. Similarly qth root of a number 'a' when

$q > 1$, (and is a rational number) means a number whose 'qth power equal 'a'

If we get $x^q = a$, then x will be called qth root of 'a'. For example:

1. $(\pm 3)^2 = 9$. Therefore square roots of 9 are 3 and -3 and $+3$ is the principal square root.

2. $(\pm 2)^4 = 16$. Therefore 4th roots of 16 are $+2$ and -2 while $+2$ is the principal 4th root.

3. $(3)^3 = 27$. Therefore 3rd root of $27 = 3$.

4. $(-3)^5 = -243$. Therefore 5th root of $-243 = -3$.

Definition: qth root of a is written as $\sqrt[q]{a}$. Here 'a' is Radicand and 'q' is index.

In radical sign "$\sqrt{\ }$" the index is 2 but convention is that it is not written.

Definition: $\sqrt[n]{a} = a^{1/n}$ and $\sqrt[n]{a^m} = a^{m/n}$.

3.4 SCIENTIFIC NOTATION

In problems of Science and Mathematics many a time we are concerned with too small or too large numbers. For example speed of light is 300,000,000 meters per second. Weight of earth is

6,000,000,000,000,000,000,000,000 kilograms.

Weight of electron is

0.000,000,000,000,000,000,000,000,000,000,910,905

kilograms. You can very well imagine as to how much difficult it is to read and write such numbers. Scientific notation comes to our help in denoting such numbers. For example :

(i) $5687 = 568.7 \times 10 = 56.87 \times 10^2 = 5.687 \times 10^3$

(ii) $250000 = 25 \times 10000 = 25 \times 10^4 = 2.5 \times 10^5$

(iii) $$0.00519 = \frac{519}{100000} = \frac{519}{10^5} = 519 \times 10^{-5}$$

$$= 5.19 \times 10^2 \times 10^{-5}$$

$$= 5.19 \times 10^{-3}$$

From the above examples we can conclude that every number can be written in the form $M \times 10^n$ when M is equal to or greater than 1 but less than 10 and n is an integer.

Now with the help of scientific notation we can write the speed of light as 3×10^8 meters per second, the weight of the earth as 6×10^{24} kilograms and the weight of an electron as 9.10905×10^{-31} kilograms.

It is easy to compare the numbers and to do the mathematical calculations by writing them in scientific notation.

For example it is clear than 4.9×10^9 is greater than 9.6×10^8 while it will not be easy to decide when the numbers are written in their usual form.

EXERCISE No. 3.3

Simplify

1. $(\frac{1}{5})^2$ 2. $(\frac{2}{13})^2$

3. $(\frac{36}{96})^2$ 4. $(-7)^2$

5. $(\frac{-3}{10})^2$ 6. $(-xy)^2$

Find out the Redicand and Index in the following:

7. $\sqrt[3]{24}$ 8. $\sqrt[4]{\frac{2}{3}}$

9. $\sqrt[6]{\frac{5}{7}}$ 10. $\sqrt[5]{ab}$

11. $\sqrt[6]{\frac{p}{q}}$

Simplify the following and write the answers only in positive exponents.

12. $3a^{-4} b^2$ 13. $\sqrt{16}\ \frac{x^{-2}}{y^{-3}}$

14. $\sqrt{8}\ x^4\ y^{-5}\ z^{-6}$ 15. $(ab^3)(a^2 b^{-4})$

16. $(3x^2)(y^{-3})^{\frac{1}{3}}$ 17. $(p^{-\frac{1}{2}})^{\frac{-2}{9}}$

18. $[(5)^{\frac{2}{7}}]^{-\frac{7}{2}}$

19. 15,000,000,000

20. 57,000,000

21. 0,000,000,000,12

22. 0,000,000,000,000,305

Write the following numbers in usual form:

23. 1.034×10^{-5} 24. 4.56×10^{7}

25. 9.421×10^{-15} 26. 4.1×10^{20}

3.5 LOGARITHMS

Definition: If in the equation $a^x = N$, a > 1 and $a \neq 1$, x is *a* real number, then x is called logarithm of N to base 'a' and is symbolically written as

$$\log_a N = x$$

In other words the equation $a^x = N$ and $\log_a N = x$ represent one and the same relation between a, x and N.

First equation is written in exponential form while the second is written in logarithmic form.

Logarithm is also defined as:

"The logarithm of any number N to the base 'a' is that exponent of a which gives us N".

In this definition we suppose that base 'a' is positive and $\neq 1$ and the exponent x is a real number i.e. we are concerned with logarithm of positive number only. We also suppose that one and only one real value of x satisfies the question $a^x = N$, i.e.,

The N will be a unique logarithm of a positive number to a given base.

For explanation see the examples below :

Exponential equation	Logarithmic equation
(1) $3^3 = 27$	(1) $\log_3 27 = 3$
(2) $10^4 = 10000$	(2) $\log_{10} 10000 = 4$
(3) $5^{-2} = 1/5^2 = 1/25$	(3) $\log_5 1/25 = -2$
(4) $a^0 = 1$	(4) $\log_a 1 = 0$
(5) $a^1 = a$	(5) $\log_a a = 1$

SOLVED EXAMPLES

Example No. 1.

Find N when $\log_3 N = 4$

Solution :

$$N = 3^4 = 81$$

Example No. 2.

If $\log_a 25 = 2$, find a.

Solution :

$\log_a 25 = 2$ means $a^2 = 25$

So $a = \pm 5$

but the base 'a' can only be +ve $\therefore a = 5$

Example No. 3.

If $\log_{10} 1000 = x$, find x.

Solution :

$\log_{10} 1000 = x$ means $10^x = 1000$

or $10^x = 10^3$

$\therefore x = 3$

EXERCISE NO. 3. 4

Write the following equations in exponential form :

(1) $\log_6 36 = 2$

(2) $\log_5 125 = 3$

(3) $\log_{10} 100 = 2$ (4) $\log_a 1 = 0$

(5) $\log_4 64 = 3$

Write the following equations in logarithmic form ;

(6) $3^4 = 81$ (7) $5^4 = 625$

(7) $2^{-4} = 1/16$ (9) $7^0 = 1$

(10) $36^{1/2} = 6$ (11) $10^5 = 100000$

Find the values of unknown a, x or N from the following equations :

(12) $\log_4 16 = x$ (13) $\log_{10} 10 = x$

(14) $\log_2 64 = x$ (15) $\log_3 N = 4$

(16) $\log_{16} N = \frac{1}{2}$ (17) $\log_{27} N = \frac{1}{3}$

(18) $\log_a 81 = 2$ (19) $\log_a 128 = 7$

(20) $\log_a 125 = 3$

3.6 LAWS OF LOGARITHM

Three of the laws of Logarithm are given below without proof. Students are not expected to know their proof. These laws are very useful specially in solving some of the very complicated problems.

First Law

Log (logarithm) of the product of two numbers equals the sum of the logs of these numbers, i.e.

$$\log_a MN = \log_a M + \log_a N$$

For example :

(i) $\log_a 5 \times 7 = \log_a 5 + \log_a 7$

(ii) $\log_a 41 \times 3 = \log_a 41 + \log_a 3$

This law can be extended to the log of product of more than two numbers.

So $\log_a MNP = \log_a M + \log_a N + \log_a P$

and $\log_a MNPQ = \log_a M + \log_a N + \log_a P + \log_a Q$.

For example :

(1) $\log_{10} 2 \times 3 \times 5 = \log_{10} 2 + \log_{10} 3 + \log_{10} 5$

(2) $\log_a (21 \times 53 \times 75 \times 87)$

$= \log_a 21 + \log_a 53 + \log_a 75 + \log_a 87$

Second Law

Log of quotient of two numbers is equal to the difference of the log of the numerator and the log of denominator, i. e.

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

e.g. $\log_a \frac{7}{3} = \log_a 7 - \log_a 3$

$\log_{10} \frac{3}{5} = \log_{10} 3 - \log_{10} 5$

Third Law

The log of power of a number is equal to the product of the exponent and the log of that number, i.e.

$$\log_a M^n = n \log_a M$$

For example :

(1) $\log_a 5^4 = 4 \log_a 5$

(2) $\log_a (65)^{125} = 125 \log_a 65$

(3) $\log_{10} (771)^{\frac{1}{5}} = \frac{1}{5} \log_{10} 771$

EXERCISE NO. 3. 5

Express the following logs in the form of sum or difference of logs :

(1) $\log_a 5 \times 7$

(2) $\log_a 4 \times 7 \times 9$

(3) $\log_a xy$

(4) $\log_a uvw$

(5) $\log_a \frac{x}{y}$ (6) $\log_a \frac{lm}{a}$

(7) $\log_a \frac{uv}{wx}$

Write the following in the form of a singular log :

(8) $\log_a 3 + \log_a 7$ (9) $\log_a x + \log_a y + \log_a z$

(10) $\log_{10} 3 + \log_{10} 4 - \log_{10} 5$ (11) $3 \log_{10} 5 + \log_{10} 5$

If $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$ and $\log_{10} 5 = 0.6990$ Find the values of the following logs :

(12) $\log_{10} 6$ (13) $\log_{10} 15$

(14) $\log_{10} 30$ (15) $\log_{10} 8$

(16) $\log_{10} 9$ (17) $\log_{10} 125$

(18) $\log_{10} \frac{10}{\sqrt{5}}$ (19) $\log_{10} \frac{10}{\sqrt{3}}$

3.7 COMMON LOGARITHM

Two systems of logarithms are commonly used. In one of them base is taken to be an irrational number 'e' which approximately equals 2.718. This system is called 'System of Natural Logarithms'. In the other system the base is taken to be 10 and it is called 'System of Common Logarithms'. The later system is more suitable for mathematical calculations. Here we shall discuss only common logarithms, therefore, if the base is not mentioned somewhere it will be 10, i.e. log m shall mean $\log_{10}$ m. Now consider the following examples :

Exponential Sentence	Equivalent Logarithmic Sentence
$10^4 = 10000$	$\log 10000 = 4$
$10^3 = 1000$	$\log 1000 = 3$
$10^2 = 100$	$\log 100 = 2$

$$10^{1} = 10 \qquad \log 10 = 1$$

$$10^{0} = 1 \qquad \log 1 = 0$$

$$10^{-1} = \frac{1}{10} = 0.\,1 \qquad \log 0.\,1 = -1$$

$$10^{-2} = \frac{1}{100} = 0.\,01 \qquad \log 0.\,01 = -2$$

$$10^{-3} = \frac{1}{1000} = 0.\,001 \qquad \log 0.\,001 = -3$$

$$10^{-4} = \frac{1}{10000} = 0.0001 \qquad \log 0.\,0001 = -4$$

These examples lead us to the following results :

(1) Logs of the integral power of 10 are integers.

(2) The logs of the numbers lying between 1000 and 10,000 lie between 3 and 4, i.e., their integral part shall be 3.

(3) The logs of the numbers lying between 100 and 1,000 lie between 2 and 3 i.e., their integral part shall be 2.

(4) The logs of the numbers lying between 10 and 100 lie between 1 and 2, i.e., their integral part shall be 1.

(5) The logs of the numbers lying between 1 and 10 lie between 0 and 1, i.e., their integral part shall be 0.

(6) The logs of the numbers lying between 0.1 and 1 lie between 0 and −1 i.e., their integral part shall be −1.

(7) The logs of the numbers lying between 0.01 and .1 lie between −1 and −2, i.e., their integral part shall be −2.

(8) The logs of the numbers lying between 0.001 and 0.01 lie between −2 and −3, i.e., their integral part shall be −3.

(9) The logs of the numbers lying between 0.0001 and 0.001 lie between −3 and −4, i.e., their integral part shall be −4.

The above results can be summed up in the following example:

Integral part of the logarithm of 9234 is 3.

Integral part of the logarithm of 923. 4 is 2.

Integral part of the logarithm of 92.34 is 1.

Integral part of the logarithm of 9.234 is 0.

Integral part of the logarithm of 0.9234 is $\bar{1}$ (−1 is designated as $\bar{1}$ and read as bar one).

Integral part of the logarithm of 0.09234 is $\bar{2}$.

Integral part of the logarithm of 0.009234 is $\bar{3}$.

In brief we conclude that the integral part of the logarithm of any number greater than unity is positive and is less by one than the number of digits to the left of the decimal point. The integral part of the logarithm of a number less than unity is negative and is greater by one than the number of zeros which follow the decimal point.

The logarithm of a number, in fact, consists of the integral part called the characteristic and a decimal part called the mantissa.

The method of finding the mantissa from log table is given below :

(i) Leave out the decimal point in the given number and make it a four digit number. Add zero/zeros if need be.

(ii) The first two figures of the number are found in the extreme left column of the table. Thus to find the mantissa of log 4859, look for 48 in the first column of the table

(iii) From 48 pass along the horizontal line to the number in the vertical column headed by the third figure (which is 5 in this example). We obtain the mantissa .6857.

(iv) The fourth figure (9 in the example) is found by using the Mean Differences Columns at the extreme right. Move horizontally from 48 in the first column to the Mean Difference column headed by 9. The mean difference for 9 is 8. Add the mean difference to the part of the mantissa as given below.

Mantissa of log 485 = .6857
Mean difference for 9 = 8
Mantissa of log 4859 = .6865

To find the complete value of log 4859, prefix the characteristic to the left of the decimal. Since the integral part of the given number consists of 4 digits, therefore, characteristic is 3.

∴ log 4859 = 3. 6865

In the same way log 48.59 = 1.6865 (There are two digits in the integral part, ∴ characteristic is 1).

log 4.859 = 0.6865 (Characteristic is 0 because there is one digit in the integral part of the number).

log .4859 = $\bar{1}$. 6865 (−1 in the characteristic is written as $\bar{1}$, one bar because characteristic is negativ not the mantissa).

log.04859 = $\bar{2}$. 6865

Example 1.

Find log 4563,

Solution:

Characteristics = 3 .

In order to find mantisa, look up 45 in the first column of the log table. From 45, pass along the horizontal line to the number in the vertical column headed by 6. We obtain the mantissa . 6590.

As already explained, we use the Mean Difference Column at the extreme right for the fourth figure which is 3 in this example. Move horizontally from 45 to the Mean Difference Column headed by 3. The mean difference for 3 is 3. Hence

Mantissa of log 456 = . 6590
Mean Difference for 3 = 3

Mantissa of log 4563 = . 6593

log 4563 = 3.6593

Example 2.

Find log 8.3

Solution:

Characteristic = 0

(1) In order to find mantissa, leave out the decimal point and make it a four digit number, i.e., 8300.

(2) Look for 83 in the first column of the table.

(3) The third digit is zero.Therefore, we find number 9191 at the intersection of the horizontal line passing through '83' and the vertical column headed by '0' . In the difference columns there is no '0'.

Hence mantissa = .9191

$\therefore$ log 8.3 = 0.9191

Example 3

Find log 5676. 8

Solution :

log 5676.8 = log 5677 (approximately)

Characteristic = 3

mantissa = . 7541

$\therefore$ log 5676. 8 = 3.7541

Note : With the help of the given logarithmic tables we can find the mantissa of the log of a four digit number only. Therefore, in order to find the mantissa of log 5676.8 we have taken log 5676.8 approximately equal to 5677. If the given number were 56.768 even then we would have found mantissa of log 5677. Of course the characteristic in the case would be 1 and not 3.

Example 4.

Find log 0.0075.

Solution :

Characteristic = $\bar{3}$.

In order to find mantissa make the number 7500.

$\therefore$ mantissa = . 8751

log 0.0075 = $\bar{3}$. 8751

By writing −3 as $\bar{3}$ we shall mean that only characteristic is −ve and not the mantissa, because the mantissa is always taken positive.

Note :— Remember that if the value of the log of some number is given as −ve, then mantissa should be made +ve. See example for explanation. Suppose the log of some number is equal to −2.5678.

Now $-2.5678 = -2 - .5678 = -2 - 1 + 1 - .5678$

$= -3 + .4322 = \bar{3}.4322$

Here the characteristic is −3 and the mantissa is +.4322. Since sign with 3 is −ve but mantissa .4322 is +ve., therefore, we cannot write it as −3.4322 We can write it as $\bar{3}$.4322.

ANTI-LOGARITHM

If the log of a number is given, the the number is obtained by using the Table of Anti-Logarithms. In the equation log N = x N is called the anti-log of x . For explanation see the example given below.

Example

If log N = 1.8256, find N.

Explanation:

Here the characteristic will be used only to determine the place of decimal point. In the anti-log table you will find number 6683 at the intersection of the line passing horizontally through .82 and the column headed by 5 . Similarly in the difference column you will find 9 under 6. Therefore, the number whose mantissa is .8256 would be 6683 + 9 = 6692.

Since the characteristic is one, therefore, there are two digits to the left of the decimal in the number whose logarithm it is .

$\therefore \quad N = 66.92$

EXERCISE NO. 3.6

Find the value of the following using the table of logarithms:-

(1) log 5 (2) log 549 (3) log 57.4
(4) log 789.2 (5) log 2798 (6) log 7.46
(7) log 457000 (8) log 0.4721 (9) log 0.0067
(10) log 0.05708

Find the anti-logs of the following :

(11) 1.9445 (12) 2.4425 (13) 3.8763
(14) 0.5537 (15) $\bar{2}.8716$ (16) $\bar{3}.7757$
(17) $\bar{2}.1476$

MATHEMATICAL COMPUTATIONS WITH THE HELP OF LOGARITHM

Example 1.

Solve 8.485 x 48.67 with the help of log.

Solution:

Suppose $x = 8.485 \times 48.67$

$\log x = \log(8.485 \times 48.67) = \log 8.485 + \log 48.67$
(Rule I of log)

$0.9287 + 1.6872 = 2.6159$

i.e. $\log x = 2.6159$

using the anti-log table, we find

$x = 413.0$

Example 2.

Find the product of 3.825 and 0.00729 with the help of logarithms :

Solution :

Suppose $x = 3.825 \times 0.00729$

$\log x = \log (3.825 \times 0.00729)$

$= \log 3.825 - \log 0.00729$

$= .5827 + \bar{3}.8627$

$= 2.4454$

Explanation of Addition

$$\begin{array}{r} 0.5827 \\ \underline{\bar{3}.8627} \\ \bar{2}.4454 \end{array}$$

+1 carried forward from decimal fraction and added to $\bar{3}$ or -3 gives us -2 or $\bar{2}$.

Using anti-log table, we find

$x = .02789$

Example 3.

$$\text{Solve} \quad \frac{23.4 \times 0.681}{4.13}$$

Solution:

$$\log N = \log \frac{23.4 \times 0.681}{4.13}$$

$= \log 23.4 + \log 0.681 - \log 4.13$

$= 1.3692 + \bar{1}.8331 - 0.6160$

$= 0.5863$

Explanation :

$$\begin{array}{r} 1.3692 \\ \underline{+\ \bar{1}.8331} \\ 1.2023 \\ \underline{-\ 0.6160} \\ 0.5863 \end{array}$$

Using anti-log table, we have

$$N = 3.858$$

Example 4

Find the value of $(5.617)^5$.

Solution :

Suppose $N = (5.617)^5$.

$$\log N = \log (5.617)^5 = 5 \log 5.617$$

$$= 5 \times 0.7495 = 3.7475$$

Using anti-log table, we have

$$N = 5591$$

Example 5.

Find the value of $\sqrt[7]{28.3}$

Solution :

Suppose $N = \sqrt[7]{28.3}$

$$\log N = \sqrt[7]{28.3} = \log (28.3)^{1/7}$$

$$= \frac{1}{7} \log 28.3 = \frac{1}{7} (1.4518) = 0.2074$$

Using table of antilogarithms, we get

$$N = 1.612$$

EXERCISE No. 3.7

Solve the following with the help of logarithms :

(1) 4.578×62.16 (2) 16.78×15.8

(3) $3.97 \times 1.45 \times 7.89$ (4) $\frac{15.57}{4.25}$

(5) $\frac{26.69}{8.31}$ (6) $\frac{8.57 \times 24.7}{7.51}$

(7) $\dfrac{12.3 \times 6.78}{2.16}$ (8) $(3.54)^6$

(9) $\sqrt{8.57}$ (10) $\sqrt{125.4}$

(11) $\sqrt[5]{12.43}$ (12) $\sqrt{\dfrac{7.24}{2.12}}$

(13) $\dfrac{(84.3)^{1/3}\sqrt{39.7}}{23.4}$ (14) $\dfrac{(6.49)^2\sqrt[8]{8.21}}{17.9}$

(15) $3.97 \times 1.45 \times 7.89$

(16) $85.7 \times 2.47 \div 8.89$

(17) $67.35 \times 48.27 \div (16.18)^2$

(18) $(84.5)^{\frac{1}{3}}(39.7)^{\frac{1}{2}} \div (23.4)^{\frac{1}{2}}$

CHAPTER 4

STATISTICS

4.1 INTRODUCTION

There is hardly any field of human inquiry where one can do without statistics. From the very beginning there has been a trend to measure quantities. In the beginning, consciously or unconsciously, the quantities and numbers were used. In the olden times the kings also used to take help from statistics in order to assess their material power whereby they were able to know as to how much expenditure was to be borne during peace and war. In this way they were able to plan better. They had a different procedure for collecting statistics but it went through a change with the passage of time.

Now-a-days you hear from the radio, T. V. or read in newspapers that last year so many tons of wheat or rice were produced and the estimate for this year is so many tons. You would also have read that the country needs so much wheat every year. All this is also related to statistics. We come to know from the history that an estimate of wealth and population was made in Egypt in 3050 B.C. Similarly European countries collected statistics of population from time to time. In 16th century many of the astronomers collected statistics regarding the rotation of planets with the help of which 'the Laws of Kepler' were discovered. Afterwards Newton verified these laws mathematically with the help of the 'law of gravitational pull'. In Europe during the seventeenth century, statistics of births, deaths and marriages were collected and were presented in the form of tables.

In the eighteenth century due to the efforts and sustained hard work of mathematicians, a deep relation between mathematics and statistics was established. The 'Theory of Probability' came into being as a result.

It can be proved by looking at the pre-Pakistan history that in India, about 300 years ago, statistics were collected in an organised way for administrative purposes. We come to know through 'Aaeen-Akbari' that in the regime of Akbar the Great, statistics were collected for administrative affairs , and partitioning of land. Similarly, statistical tables of prices and crops were prepared in the reign of Sher Shah Soori.

Through this short historical background we come to know the importance and usefulness of statistics. In the present age, statistics has got a great direct concern with our practical life. Its use is rapidly increasing in medicine, Science and Engineering Besides this it has become a necessity to make use of statistics in business, agriculture, banking, insurance, commerce, education, research, civil and defence institutions of the government and in the Departments of Meteorology, Irrigation and Food etc. Collection of data is an important factor in the use of statistics.

4.2 COLLECTION OF DATA

Before some development scheme or budget is prepared, the concerned data are collected. Every year the Government prepares budget estimates for the coming year on the basis of the data of the previous year. It is through these data that we come to know as to how much did the Government spend on education food, defence , etc., in the past year and what funds have been provided for these items in the budget of the current year. Similarly the teachers assess the ability of their students on the basis of their scores, and it is on the basis of these scores that the academic performance of a school is judged.

The following points are to be kept in view in the collection of data otherwise there is a probability of wrong conclusions to be drawn :

The data collectors :

(i) should be industrious, intelligent and reliable.

(ii) should be properly trained,

(iii) should avoid heresay and guess work.

4.3 UNGROUPED DATA

Sometimes the data to be collected are ungrouped or without any order and sequence. For example, the data given below are ungrouped.

Ten students took a test in General Mathematics and scored the following marks:

25,30, 10, 1, 6, 18, 15, 30, 6, 20.

4.4 GROUPED DATA

We cannot get the requisite information with ease and facility by looking at the ungrouped data but if these data are grouped in a specific way we can get some of the results even at a cursory glance. We shall see in the following article how the data are classified.

4.5 CLASSIFICATION OF DATA

This is the way in which the data are classified into groups of the same kind. For example, the data given in article 4.3 can be classified in the following manner:

Marks obtained.	Number of Students.
1 – 10	4
11 – 20	3
21 – 30	3

4.6 TABULATION OF DATA

Representation of statistical data in classified form in rows and columns, is called 'Tabulation of Data'.One of the advantages of tabulation is that the scattered data are represented in a compact and clear form wherefrom results can be easily deduced.

The data mentioned above can be tabulated in the following way :

TABLE 1

Marks obtained.	Number of students.
1 – 10	4
11 – 20	3
21 – 30	3

4.7 CLASS INTERVAL

In table I (1 – 10),(11 – 20)and (21–30) are class intervals.

4.8 CLASS FREQUENCY (in Table 1)

The class frequency from 1 to 10 = 4
The class frequency from 11 to 20 = 3
The class frequency from 21 to 30 = 3

i.e. the number of units in a class interval is called 'Class Frequency'.

4.9 LOWER LIMIT AND UPPER LIMIT (in Table 1)

The lower limit in class interval (1 – 10) is 1 and the upper limit is 10.

The lower limit in class interval(11 – 20) is 11 and the upper limit is 20.

The lower limit in class interval (21 – 30) is 21 and the upper limit is 30.

Note : In every class interval the number on the left is 'lower limit' and number on the right is "upper limit'.

4.10 MAGNITUDE OF CLASS INTERVAL

The difference of first two lower limits or first two upper limits is called 'magnitude of class interval'. In Table 1.

The difference of first and second lower limits

= 11 – 1 = 10

The difference of first and second upper limits

= 20 – 10 = 10

∴ The difference of first two lower limits = the difference of two upper limits.

It is easy to see that in a table, the difference of any two consecutive lower limits is always equal to the difference of any two consecutive upper limits.

Hence the difference of any two consecutive lower limits or upper limits is called 'magnitude of class interval.

4.11 FREQUENCY DISTRIBUTION

The set of frequencies associated with class intervals is called 'Frequency Distribution'. The frequency of each class interval is written against it. The following example illustrates the 'Frequency Distribution'.

Example:

50 scouts of a school went to Murree. There were 8 scouts whose ages were from 10 to 11 years, 25 students whose ages were from 12 to 13 years, 15 students whose ages were from 14 to 15 years and 2 students whose ages were from 16 to 17. Prepare a frequency distribution table and write the class frequencies.

FREQUENCY DISTRIBUTION

Ages (in years)	Numbers of Scouts
10 – 11	8
12 – 13	25
14 – 15	15
16 – 17	2

Frequency for the class interval (10 – 11) = 8
Frequency for the class interval (12 – 13) = 25
Frequency for the class interval (14 – 15) = 15
Frequency for the class interval (16 – 17) = 2

4.12 TO PREPARE FREQUENCY DISTRIBUTION TABLE FROM UNGROUPED DATA

(1) TALLY METHOD : The grouping of data becomes easier with the use of this method. Under this method the ungrouped data are divided into classes. In these classes it is one's own choice to select the magnitude of class interval in case it is not already given in the problem. Usually the magnitude of class intervals is taken as 2, 5, 10, 15, or 25. The steps to be remembered in tally method are as under:

(i) First of all prepare the following three columns:
(a) Class intervals (b) Tally marks (c) Frequencies.

(ii) Pick out the greatest and the least numbers from ungrouped data and determine the number of classes according to the magnitude of class intervals selected by you.

(iii) See in which class interval each element of the ungrouped data falls. Draw a small tally mark (/) against that interval and also tick the element concerned with a sign " √ " so that you can remember that you have accounted for that element. If some element is repeated 4 times in a class interval, make four tally marks like (/ / / /) and if the same is repeated five times then make the tally marks (~~////~~) and if some element is repeated 8 times, make the tally marks like (~~////~~ / / /).

(iv) Having ticked all the elements, count the number of tally marks against each class interval and write that number in the frequency column against that class interval.

See the following examples for illustration.

Example 1

Following are the scores of 10th class students in a practical examination out of 50 marks :

0, 1, 3, 5, 4, 18, 8, 23, 49, 11, 36, 45, 40, 35, 40, 34, 25, 17, 10, 9, 28, 35, 49, 2, 16.

If 10 be the magnitude of class interval prepare a frequency distribution table using tally method.

Solution :

0, 1, 3, 5, 4, 18, 8, 23, 49, 11, 36, 45, 40, 35, 40, 34, 25, 17, 10, 9, 28, 35, 49, 2, 16.

Highest score = 49

Lowest score = 0

Magnitude of class interval = 10

FREQUENCY DISTRIBUTION TABLE

Class intervals	Tally marks	Frequency
0 – 9	~~////~~ ///	8
10 – 19	~~////~~	5
20 – 29	///	3
30 – 39	////	4
40 – 49	~~////~~	5
	Number of tally marks = 25	Sum of frequencies= 25

Note : Remember, that the magnitude of class interval (10 – 19) is 10. In the preceding example the class interval can also be taken as follows :

(1) 0 – 10
(2) 10 – 20
(3) 20 – 30
(4) 30 – 40
(5) 40 – 50

While tabulating such class intervals it is important to note that no class interval includes its upper limits.

Upper limit 10 of class interval (0 – 10) is not included in it.

Upper limit 20 of class interval (10 – 20) is not included in it.

Upper limit 30 of class interval (20 – 30) is not included in it.

Upper limit 40 of class interval (30 – 40) is not included in it.

Upper limit 50 of class interval (40 – 50) is not included in it.

Following is the Frequency Distribution according to the above class intervals :

Class intervals	**Tally marks**	**Frequencies**
0 – 10	卌 |||	8
10 – 20	卌	5
20 – 30	|||	3
30 – 40	||||	4
40 – 50	卌	5
	Number of tally marks = 25	Sum of frequencies = 25

Example 2

20 persons work in an office. Their monthly salary (in rupees) is 275, 250, 280, 240, 260, 275, 280, 250, 240, 270, 275, 245, 250, 265, 270, 275, respectively. Prepare a frequency distribution table by fixing 15 as magnitude of class-intervals.

Solution :

FIRST METHOD :

FREQUENCY DISTRIBUTION

Class intervals	Tally marks	Frequencies
240 – 255	250, 240, 250, 245, 250, 240	6
255 – 270	260, 268, 265, 265, 255	5
270 – 285	275, 280, 275, 280, 270 275, 275, 275, 270	9
	Total entries = 20	Number of persons = 20

SECOND METHOD : (Using Tally Marks)

275, 250, 280, 240, 260, 275, 280, 250, 240, 270, 275, 268, 255, 265, 275, 245, 250, 265, 270, 275.

FREQUENCY DISTRIBUTION

Class interval	Tally marks	Frequencies
240 – 255	~~////~~ /	6
255 – 270	~~////~~	5
270 – 285	~~////~~ ////	9
	Number of tally marks = 20	Sum of frequencies=20

Note · Students should prepare the frequency distribution table with the following intervals :

(i) 240 – 254

(ii) 255 – 269

(iii) 270 – 284

EXERCISE NO. 4.1

1. Answer the following questions using the given table :

Marks obtained.	Number of Students
1 – 15	5
16 – 30	23
31 – 45	18
46 – 60	11
61 – 75	3

(i) How many class-intervals are there ?

(ii) What is the lower limit in the interval (16 – 30) ?

(iii) What is class frequency of (46 – 60) ?

(iv) What is the number of students securing marks from 31 to 60 ?

(v) What is the magnitude of class-interval (16 – 30) ?

(vi) What is the upper limit of class-interval having 18 as class frequency ?

(vii) What is the magnitude of class-interval in the given table?

(viii) In which class-interval does the least number of students lie ?

(ix) What is the number of students securing mark from 1 to 75 ?

2. Prepare a frequency distribution table with the help of the following data:-

8, 15, 25, 16, 12, 9, 1, 23, 14, 21, 4, 22, 11, 3, 2, 5, 7, 17, 25, 13. Use the magnitude of class interval as 5 and find the number of class-intervals.

3. The weights (in kilograms) of the players of a school are as follows :

50, 51, 52, 58, 57, 55, 54, 56, 56, 58, 52, 55, 53, 58, 56, 53, 53, 52, 54, 57, 52, 54, 53, 52.

Using tally method prepare the frequency distribution table with a class interval of 2 kilograms. Also find the class frequency of the class-interval (52 – 53).

4. A group of 10th class students secured the following marks in a practical examination :

35, 40, 45, 38, 45, 35, 49, 48, 45, 3, 5, 14, 37, 42, 41, 20, 14, 1, 9, 46, 8. Prepare a frequency distribution table taking 10 as magnitude of class interval. From that table find the class-interval with the least class frequency.

5. The students with the following roll numbers were present in a class when the roll was called:-

1, 2, 4, 5, 7, 8, 9, 10, 11, 12, 16, 17, 18, 20, 21, 23, 25, 26, 28, 29, 30, 34, 36, 38, 39, 43, 44, 46, 48, 50, 52, 57, 58, 60.

Prepare a frequency distribution table with the groups of tens and find, in which group the attendance was the best ?

6. In a factory the monthly salary (in rupees) of 40 workmen is :

810, 850, 870, 820, 880, 890, 900, 898, 899, 815, 826, 837, 848, 859, 861, 872, 882, 889, 809, 840, 835, 801, 802, 817, 845, 850, 835, 854, 840, 900, 865, 868, 835, 875, 842, 832, 825, 876, 874, 838.

Taking 15 as magnitude of class-interval, prepare a frequency distribution table and find the number of workmen drawing

Rs. 831 to 845.

7. Consult the given table to tell, which of the following statements are true and which of them are false ? Write the false statements in a correct form :

Salary of employees (in rupees)	Number of employees
400 – 420	11
420 – 440	16
440 – 460	65
460 – 480	12
480 – 500	35
500 – 520	20

(i) The lower limit in (400 – 420) is 420

(ii) The upper limit 440 is included in (420 – 440).

(iii) The number of employees drawing from Rs.400 to Rs.520 is 159.

(iv) The class frequency is the greatest in class interval (440 – 460).

(v) The interval with the least class frequency is (460 – 480).

8. Prepare a frequency distribution table with the following data when (0 – 8), (8 – 16) etc. are the class-intervals :

5, 40, 47, 0, 4, 11, 8, 16, 24, 32, 40, 0, 1, 23, 35, 28, 23, 16, 6, 12, 10, 11, 25, 29, 30, 31, 45, 44, 31, 34, 14, 15, 20, 19, 18, 17, 25, 26, 30, 11, 46, 16, 9, 8, 30, 31.

4.13 MEASURES OF CENTRAL TENDENCY

AVERAGE : A single quantity, which is the representative of all the given quantities, is called an 'Average'.

KINDS OF AVERAGE : (i) Mean (ii) Median, (iii) Mode

Mean is of three kinds :

(i) Arithmetic Mean (ii) Geometric Mean, (iii) Harmonic Mean.

We shall introduce ourselves here, only with Arithmetic Mean, Median and Mode.

4.14 ARITHMETIC MEAN

This explains the central tendency of a group.

(i) Method of finding Arithmetic Mean of ungrouped data.

Arithmetic Mean is obtained by "adding all the values of variables in an ungrouped data and then dividing by the total number of such values."

Note : The quantities which keep on varying with regard to their values are called 'variables'. They are usually represented by x, y, z, etc. For example if x is a variable and its 3 values are 2, 3, 4, then the 'Arithmetic Mean, of these values is found as follows:

$$\text{A. M.} = \frac{2+3+4}{3} = \frac{9}{3} = 3$$

Hence if x is a varaible and $x_1, x_2, x_3, \ldots\ldots x_n$ are some of its values, then the **Arithmetic Mean** of these values will be as under :

$$\text{A. M.} = \frac{x_1 + x_2 + x_3 \ldots\ldots\ldots x_n}{n}$$

The formula can also be written as

$$\text{A. M.} = \frac{\Sigma x}{n}$$

'Σ' is a Greek symbol which is read as 'Sigma' and is used to denote the sum. n is the number of values of the variables x.

The method is illustrated in the following example :

Example 1 :

Abrar played 12 cricket matches for his school and got the colour of the best player by scoring 50, 85, 98, 55, 78, 82, 65, 75, 80, 72, 70, 90 runs respectively . Find the Arithmetic Mean of his runs.

Solution :

$$\text{A. M.} = \frac{50+85+98+55+78+82+65+75+80+72+70+90}{12}$$

$$\frac{900}{12} = 75$$

Hence the A. M. of Abrar's scores = 75 runs.

(ii) **Method of finding the Arithmetic Mean when frequency is given.**

In such a condition :

(i) Multiply each value of the variable by its given frequency;

(ii) Add the products ; and

(iii) Divide the sum by the total number of values of the variable.

For example :

Suppose x is a variable.

$x_1, x_2, x_3 \ldots\ldots\ldots\ldots x_n$ are its values and $f_1, f_2, f_3 \ldots\ldots, f_n$ are their corresponding frequencies respectively, then

$$\text{A. M.} = \frac{f_1 x_1 + f_2 x_2 + f_3 x_3 + \ldots\ldots + f_n x_n}{f_1 + f_2 + f_3 + \ldots\ldots + f_n}$$

This can also be written as

$$\text{A. M.} = \frac{\Sigma f x}{\Sigma f}$$

when ' f ' stands for frequency.

The following example illustrates the method.

Example 2

45 labourers work in a factory. Their daily wages are given below. Find the Arithmetic Mean of their wages.

Wages (in rupees)	Number of labourers
10	4
15	20
20	5
25	6
30	10

Solution : By representing wages by x and the number of labourers by f, we prepare the following table :

Wages (in rupees) x	Number of labourers f	f x x
10	4	4 x 10 = 40
15	20	20 x 15 = 300
20	5	5 x 20 = 100
25	6	6 x 25 = 150
30	10	10 x 30 = 300
	Σ f = 45	Σ f x x = 890

$$\text{A. M.} = \frac{\Sigma f \times x}{\Sigma f} = \frac{890}{45} = 19.78$$

Hence the A. M. of labourers,wages is 19.78 rupees.

(iii) **Method of finding Arithmetic Mean of grouped data :**

Suppose $x_1 . x_2 . x_3$ .... x_n are the central values of class-intervals and f_1, f_2, f_3, ..., f_n respectively are frequencies of these values, then

$$\text{A. M.} = \frac{f_1 \times x_1 + f_2 \times x_2 + f_3 \times x_3 + ...f_n \times x_n}{f_1 + f_2 + f_3 + ...+ f_n}$$

$$= \frac{\Sigma f \times x}{\Sigma f}$$

Remember the following three important points to find the A. M. of a grouped data :

(i) Find the central value x of the class - interval.

(ii) Find f x x .

(iii) Divide Σ f x x by Σ f (the sum of frequencies).

Example 3.

There are 700 students in a school. Classification of their ages is given below. Find the arithmetic mean of their ages.

Ages (in years)	Number of students
9 – 11	200
11 – 13	225
13 – 15	175
15 – 17	100

Solution :

Ages (in years)	Central value of the class interval. x	Number of students f	f x x
9 – 11	$\frac{9+11}{2} = 10$	200	200 x 10 = 2000
11 – 13	$\frac{11+13}{2} = 12$	225	225 x 12 = 2700
13 – 15	$\frac{13+15}{2} = 14$	175	175 x 14 = 2450
15 – 17	$\frac{15+17}{2} = 16$	100	100 x 16 = 1600
		Σf= 700	Σf x x =8750

$$\text{A. M.} = \frac{\Sigma f \times x}{\Sigma f} = \frac{8750}{700} = 12.\ 5 \text{ years}$$

Hence A. M. of the ages of 700 students is **12 years and 6 months.**

(vi) Method of finding Arithmetic Mean through a short cut method.

Usually, this method is used when the values of variables are very big. This saves time and to some extent gives relief from finding lengthy products.

Suppose $x_1, x_2, x_3, x_4, \ldots\ldots, x_n$ are the central values of class-intervals and $f_1, f_2, f_3, \ldots\ldots, f_n$ respectively are their corresponding frequencies. Choose any provisional mean from these central values and let that central value be A. (Usually the value with the highest frequency is taken as provisional mean, though it is not necessary).

Subtract this provisional mean from the central values to find 'deviations'. Denote these deviations by $D_1, D_2, D_3, \ldots\ldots, D_n$.

Find the products of respective deviations and frequencies. Add these products to get

$$f_1 \times D_1 + f_2 \times D_2 + f_3 \times D_3 + \ldots\ldots\ldots\ldots f_n \times D_n = \Sigma f \times D$$

Divide $\Sigma f \times D$ by Σf and add the provisional mean to this result (quotient). This is the required A. M.

Note : The deviation will be as under:

$D_1 = x_1 - A$

$D_2 = x_2 - A$

$D_3 = x_3 - A$

....

....

....

$D_n = x_n - A$

Remember the sequence of the following important steps for tabulation to find Arithmetic Mean through short cut method.

(1) First Column : Writing Class-intervals.

(2) Second Column : Writing given frequencies.

(3) Third Column : Finding central values (x) of the class intervals.

(4) Fourth Column : Subtracting provisional mean from the central values to find D.

(5) Fifth column : Finding f x D.

Hence A. M. $= \dfrac{\Sigma f \times D}{\Sigma f} + \text{P. M.}$

The Arithmetic Mean will be computed by the following formula :-

$$\text{A. M.} = \frac{\Sigma f \times D}{\Sigma f} + \text{P. M. (P. M. = Provisional Mean)}$$

$\Sigma f \times D$ = The sum of the products of central values and their corresponding frequencies.

Σf = The sum of the frequencies.

See the following example for illustration :

Example 4 :

Solve the problem in Example 3 with the short cut method. Take 12 as P. M.

Class Interval.	No. of Students (f)	Central value (x)	D $x - 12$	f x D
9 – 11	200	$\frac{9+11}{2} = 10$	10 – 12 = –2	200 x (2) ÷ 400
11 – 13	225	$\frac{11+13}{2} = 12$	12 – 12 = 0	225 x 0 = 0
13 – 15	175	$\frac{13+15}{2} = 14$	14 – 12 = 2	175 x 2 = 350
15 – 17	100	$\frac{15+17}{2} = 16$	16 – 12 = 4	100 x 4 = 400
	$\Sigma f = 700$			$\Sigma f \times D = 350$

Now P. M. (Provisional Mean) = 12

$$\Sigma f \times D = 350$$

$$\Sigma f = 700$$

$$\therefore \quad A.M. = \frac{\Sigma f \times D}{\Sigma f} + P.M. = \frac{350}{700} + 12$$

$$= .5 + 12 = 12.5$$

Hence A. M. of ages **is 12 years and 6 months.**

EXERCISE NO.4.2

1. Find the Arithmetic Mean :

(i) 2, 3, 4, 5, 6, 7, 8, 9, 10.

(ii) 39, 28, 12, 36, 41, 18, 20, 37, 46, 43.

(iii) 3005, 3007, 3004, 3008, 3015, 3003.

(iv) 4.3, 8.9, 6.4, 12.9, 18.7.

2. There are 8 sections of ninth class in a school. The number of students in them is 56, 46, 65, 53, 60, 64, 59 and 61 respectively. Find A. M. of the number of students in these sections.

3. Nasir scored 3, 4, 6, 3, and 2 goals respectively in 6 matches of football. Find their A. M.

4. Find Σf, x, Σfx, D and $\Sigma f \times D$ in the form of a table from the given grouped data(x stands for central value, D for deviation and f for frequency).

Marks obtained.	Number of students.
1 – 5	10
6 – 10	12
11 – 15	14
16 – 20	16

5. 90 workmen work in a factory. The detail of their daily wages is given below. Find the A. M. of their wages.

wages in (Rupees)	; x	12	15	18	21	24	27
Number of labourers	: f	10	20	30	5	15	10

6. Find the A. M. in each of the following :

(i) 7, 9, 0, −9, −7.

(ii) 8, 9, 10, 0, −9, −6.

(iii) −110, −100, 0, 80, 30, 10.

7. The classification of ages of 22 teachers of a school is given below. Find A. M. of their ages.

Ages (in years)	Number of teachers.
26 – 30	8
31 – 35	4
36 – 40	3
41 – 45	5
46 – 50	2

8. The marks obtained by the students in an examination in the subject of Science are given in the table below. Find the A. M. of their marks.

Marks obtained	Number of students
21 – 25	80
26 – 30	125
31 – 35	75
36 – 40	215
41 – 45	85
46 – 50	20

9. In a factory the labourers get their wages according to the schedule given below. Find the A. M. of their wages by

short cut method.

Wages (in rupees)	Number of labourers
310 – 330	135
330 – 350	213
350 – 370	136
370 – 390	416

10. The measurments (in decimeters) of the sizes of the children of a school are given in the table below. Find the A. M. of their sizes.

Size (in decimeters)	Number of Children
6. 1 – 6.5	100
6. 6 – 7.0	300
7. 1 – 7.5	400
7. 6 – 8.0	500
8. 1 – 8.5	100

4.15 MEDIAN

To find the median of ungrouped data

When the numbers in a data are written in ascending or descending order, the middle number is called the median of such data.

Example 1.

The members of the cricket team of a school made 18, 12, 0, 25, 20, 15, 11, 10, 25, 12, and 7 runs respectively. Find the median of their runs.

Solution :

Write the runs in ascending order :

Serial number of players	1	2	3	4	5	6	7	8	9	10	11
Runs in ascending order	0	7	10	11	12	12	15	18	20	25	25,
		5 terms				12 is median			5 terms		

The middle term is 12 and this is the median.

Example 2

8 students obtained 48, 16, 10, 45, 12, 36, 45, and 32 marks respectively in Arabic. Find their median.

Solution

First we write the marks in ascending order :

Sr. No. of the student.	1	2	3	4	5	6	7	8
Marks in ascending order	10	12	16	32	36	45	45	48
		3 terms			2 terms		3 terms	

Now there are two middle terms. i.e, 32 and 36. To find the median is to find A.M. of 32 and 36, which is

$$\frac{32 + 36}{2} = \frac{68}{2} = 34$$

$\therefore$ Median = 34

Note : We can conclude that if the number of terms is even, we shall get two middle terms and the median of all the given terms will be equal to the A.M. of the two middle terms.

4.16 MODE

Mode is the number that appears most frequently in a given data. For example, 40 students took an examination when the result was tabulated, the number of students who secured 25 marks appeared most frequently. Therefore 25 is the mode.

Example 1

Find the mode of 6, 8, 5, 6, 4, 6, 8, 6, 7.

Solution :

First we write the numbers in ascending order.

Serial No.	1	2	3	4	5	6	7	8	9
Ascending order of numbers	4	5	6	6	6	6	7	8	8
			$\underbrace{\qquad\qquad\qquad}_{\text{4 times}}$						

6 is repeated more than any other number.

$\therefore$ Mode = 6.

Example 2

12 persons work in a General Store. Their monthly wages in rupees are 350, 380, 375, 345, 380, 370, 365, 350, 348, 365,340 and 380. Find the mode of their wages.

Solution:

First we write the wages in ascending order.

Serial No.	Wages in Ascending order
1	340
2	345
3	348
4	350
5	350

6	365
7	365
8	370
9	375
10	380
11	380
12	380

380 is repeated most. The mode is , therefore 380.

EXERCISE NO. 4.3

1. Find the median in each of the following :

(i) 2, 5, 1, 4, 3.
(ii) 2, 1, 6, 5, 3, 4.
(iii) 21, 32, 48, 47, 48, 23, 10.
(iv) 3, 2, 1, 0, −1, −2, −3.

2. According to the 1981 census the population of 10 villages is given below. Find their median

407, 594, 672, 522, 733, 777, 1490, 2488, 391, 387.

3. 53 boys are standing in a row which boy is their median ?

4. The number of attendances of a student in 6 months is 36, 46, 48, 20, 47, 45 respectively. Find the median of his attendances.

5. Find the mode of 1, 3, 6, 3, 5, 4, 3, 1, 4, 3.

6. The members of cricket team of a school made the following runs. Find their mode.

10, 8, 0, 25, 13, 2, 0, 1, 0, 25, 6.

7. 8 persons work in a general store whose monthly wages (in rupees) are **800, 750, 725, 800, 800, 750, 800, 840.** Find the mode of their wages.

8. There is a six-storey building. Its storeys contain 40, 50, 50, 55, 55, 55 rooms respectively. Find the mode of rooms.

MISCELLANEOUS EXERCISE 4.4

1. In a Secondary School Board Examination 50 students obtained the following marks, respectively in Mathematics :

18, 15, 85, 63, 80, 75, 23, 10, 0, 93, 41, 0, 99, 100, 23, 66, 43, 48, 50, 60, 32, 33, 67, 97, 86, 77, 39, 16, 5, 14, 41, 51, 63, 70, 21, 11, 55, 58, 60, 98, 99, 95, 64, 13, 15, 30, 60, 45, 15, 28.

Prepare frequency table taking 15 as magnitude of class-interval.

2. Measures of the sizes (in decimeters) of 100 scouts are given below. Find the A. M. of their sizes.

Measures of sizes (in decimeters)	Number of scouts
8 – 10	5
10 – 12	18
12 – 14	42
14 – 16	27
16 – 18	8

3. The A. M. of 10 numbers is 32. Find their sum.

4. 30 students appeared in an examination. 10 students obtained 70 marks each, 8 obtained 45 marks each and the rest of the students obtained 65 marks each. Find the mode of their marks.

5. The population of different localities of a city is 700, 600, 650, 550, 760, 545, 670, 750, 700, 540. Find the median of the population of the localities.

6. Prepare the frequency distribution table of the following data taking 2 as magnitude of class-intervals. Also find the A. M. and the mode.

70, 68, 75, 67, 71, 68, 68, 74, 71, 68, 67, 70, 77, 76, 69, 67, 66, 72, 73, 74, 71, 68, 69, 67, 76, 69, 77, 70, 77, 67, 74, 67, 68, 68, 68.

7. The classification of the stature (in decimeters) of the students of a school participating in a tournament is given below. Find the A. M. of their statures.

Stature (in decimeters)	Number of Students
60.0 – 60.4	50
60.4 – 60.8	115
60.8 – 61.2	30
61.2 – 61.6	60
61.6 – 62.0	75
62.0 – 62.4	130

8. The A. M. of the ages of 25 girls is 14 years and 2 months. Find the sum of their ages.

9. The schedule of wages of 50 labourers of a factory is given below. Find the A. M. of their wages.

Wages (in rupees)	Number of labourers
900 — 910	4
910 — 920	25
920 — 930	6
930 — 940	15

10.

Weight (in grams)	Number of students
51 – 54	150
54 – 57	75
57 – 60	300
60 – 63	125
63 – 66	50

With the help of above table tell which of the following statements are true and which of them are false ? Also correct the false statements.

(i) The table is divided into 5 class-intervals.
(ii) The lower limit of (57 – 60) is 60.
(iii) The class-frequency of (60 – 63) is not 125.
(iv) The central value of (51 – 54) is 52.
(v) The difference of two consecutive lower limits (54 and 57) is 3.
(vi) The number of students in (54 – 60) is 300.
(vii) The class-frequency of (63 – 66) is 50.
(viii) The number of students in (51 – 66) is 600.
(ix) Theupper limit 63 in included in (60 – 63).
(x) The class-interval with the least class-frequency is (54 – 57).

CHAPTER 5

RATIO, PROPORTION, INHERITANCE AND PARTNERSHIP

5.1 RATIO

In our daily life, we generally compare two homogeneous quantities in two ways. Sometimes we take their difference and state as to how much smaller or greater is one quantity than the other e.g., Aslam's age is 12 years and Hamid is 14 years old. We can say that Aslam is two years younger than Hamid or Hamid is two years older than Aslam.

Very often the comparison between two homogeneous quantities is expressed as the quotient of these quantities e.g. if one book is priced at Rs.8 and the other at Rs.4, the first book costs double the second. In other words

$$8 \div 4 = \frac{8}{4} = \frac{2}{1}$$

We express this relationship as 2 : 1 and read it as the ratio 'two to one'. It is clear that in the horizontal way, the ratio is written as 2 : 1 and in the form of a common fraction, it is written as $\frac{2}{1}$. It is to be noted that $\frac{2}{1}$ or 2 : 1 is a number and to add the word 'books' or 'rupees' to it is incorrect.

According to the golden rule of fractions, if we multiply numerator and denominator by the same quantity, the resulting fraction is equivalent to the original ratio e.g., ratio 2 : 3 = fraction $\frac{2}{3}$.

and

$$\frac{2}{3} = \frac{2 \times 2}{3 \times 2} = \frac{2 \times 3}{3 \times 3} = \frac{2 \times 4}{3 \times 4} = \frac{2 \times 5}{3 \times 5} = \ldots\ldots\ldots\ldots$$

$$\left\{ \frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \frac{10}{15} = \ldots\ldots\ldots \right\}$$ is the set of fractions equivalent to the ratio 2 : 3.

The set of their equivalent ratios can be written as

{ 2 : 3, 4 : 6, 6 : 9, 8 : 12, 10 : 15, ... }

It is clear that to find any member of this set, we multiply the parts of the given ratio (2 : 3) by the same number.

From the above discussion we also infer that

18 : 14 = 9 : 7 [Dividing the parts of the ratio by the same number

Example 1

Find the ratio between Rs.1.50 and Rs.2.25.

Solution :

Required ratio = 1.50 : 2.25.

$= 1\frac{1}{2} : 2\frac{1}{4}$

$= \frac{3}{2} : \frac{9}{4}$

$= \frac{3}{2} \times 4 : \frac{9}{4} \times 4$ (By multiplying with the L.C.M. of 2 and 4)

= 6 : 9

= 2 : 3 (dividing the terms by 3)

Example 2

What is the ratio between 3 kilograms 5 hectograms and 5 kilograms ?

Solution:

3 kilograms 5 hectograms = 3.5 kilograms
5 kilograms = 5.0 kilograms
Hence the required ratio = 3.5 : 5.0
= 35 : 50 (multiplying the terms by 10)
= 7 : 10 (dividing the terms by 5).

Example 3.

Zaid earned Rs.15 and spent Rs.4.50. Bakar spent Rs.2 out of an income of Rs.12. Who spent more with regard to his income ?

Solution :

Ratio between the expenditure and income of Zaid.

Expenditure : Income

$$4\tfrac{1}{2} : 15$$

$$= \frac{9}{2} : 15$$

$$= 9 : 30 \quad [\text{ Multiplying the terms by } 2\ ]$$

$$= 3 : 10 \quad [\text{ Dividing the terms by } 3\ ]$$

Or

$$\text{Zaid's } \frac{\text{Expenditure}}{\text{Income}} = \frac{3}{10}$$

Ratio between the expenditure and income of Bakar.

Expenditure : Income

$$2 : 12$$

$$= 1 : 6 \quad (\text{Dividing the terms by } 2\ ]$$

$$\text{Or} \quad \text{Bakar's } \frac{\text{Expenditure}}{\text{Income}} = \frac{1}{6}$$

The expenditure-income ratios of Zaid and Bakar in fractions are $\frac{3}{10}$ and $\frac{1}{6}$ respectively. If the denominators of the fractions are made equal in both cases, the expenditures will be easy to compare.

L. C. M. of 10 and 6 is 30. We will ,therefore, make the denominators of each fraction 30.

$\frac{\text{Expenditure}}{\text{Income}}$ of

(i) Zaid $= \frac{3}{10} = \frac{3}{10} \times \frac{3}{3} = \frac{9}{30}$

(ii) Bakar $= \frac{1}{6} = \frac{1}{6} \times \frac{5}{5} = \frac{5}{30}$

$$\frac{9}{30} > \frac{5}{30}$$

∴ Zaid spent more with regard to his income.

EXERCISE NO. 5.1

Find the ratio between the quantities (in the given order):

1. 35 books and 42 books.
2. Rs. 11.50 and Rs.6.75.
3. 3.5 meters and 5 meters.
4. 4 kilometers 70 meters, 28 kilometers 49 decameters.
5. 2 kilograms 7 hectograms, 3 kilograms 6 hectagrams
6. $2\frac{1}{2}$, $3\frac{3}{4}$

7. Hameed spent Rs.13.50 out of an income of Rs.20, and Iqbal spent Rs.18 out of an income of Rs.32. Who spent more with regard to his income ?

8. Wasim spent Rs.4.50 out of Rs.6 and Naeem spent Rs.8 out of Rs.12. Who spent more ?

9. Raheem secured 360 marks out of 800 in the quarterly examination and 425 marks out of 900 in the six-monthly examination. Did he progress or degress ?

10. A bus covered a distance of 90 kilometers in 3 hours and a train covered a distance of 124 kilometers in 4 hours. Find the ratio between per hour speeds of the bus and the train.

5.2 PROPORTION

You know that equality of two ratios is called "Proportion". A proportion has four terms. The first and the fourth terms are called "Extremes" while the second and the third terms are called "Means" and

Product of extremes = Product of means.

Given any three terms of a proportion, we can find the fourth with the help of above mentioned property.

Example 1.

Find the replacement for the frame □ in the proportion.

4 : 10 = 6 :

Solution :

Since in a proportion, the product of means equals the product of extremes

$$\therefore \quad 4x = 10 \times 6$$

$$\text{or} \quad \frac{4x}{4} = \frac{10 \times 6}{4}$$

$$\text{or} \quad = 15$$

5.3 PROPORTION BETWEEN TWO SETS

Order of the terms in a ratio or proportion is very important 5 : 4 and 4 : 5 are two different ratios.

A proportion determined by four numbers given in a specific order is a particular form of proportion between two sets i.e. the ratio between all the corresponding pairs is the same. Hence 'Direct proportion' between two sets means that in a given order, the ratio between the corresponding pairs, according to the established correspondence, is the same.

The symbol ' $\propto$ ' is used for 'direct proportion'.

An important property of direct proportion between two sets of positive numbers is that if the elements of one increase in magnitude the elements of the other set also increase in magnitude and if the elements of one set decrease in magnitude then so do the elements of the other set in the same ratio.

Some examples involving direct proportion are solved below:-

Example 1

The price of 7 small chairs is Rs.140. Find the price of 11 such chairs.

Solution

Suppose the required price is Rs. x.

Since the number of chairs has increased, therefore, the price will also increase. So it is a problem of direct proportion.

or $\frac{140}{x} = \frac{7}{11}$ $\therefore\ 7x = 140 \times 11$

$$x = \frac{140 \times 11}{7} = 220$$

Hence the required price = Rs. 220

Alternate Method

Suppose the required price is x.

Chairs	Price
7	Rs. 140
11	Rs. x

The ratio between x and 140 is the same as the ratio between 11 and 7.

Hence $\frac{x}{140} = \frac{11}{7}$ or $x = \frac{11 \times 140}{7} = 220$ rupees

Example 2

The price of one dozen oranges is Rs.6. How many oranges can be bought for Rs. 9 ?

Solution.

Suppose required number of oranges is $= x$

Since Rs. 9 are more than Rs.6. Therefore, more oranges can be bought than before.

ordered $\{x, 12\} \propto$ ordered $\{9, 6\}$

$$\frac{x}{12} = \frac{9}{6}$$

or $x = \frac{9}{6} \times 12 = 18$

Alternate Method

Price	Number of oranges
Rs. 6	12
Rs. 9	x

With more money available, more oranges can be purchased. There is, therefore, Direct Proportion between the amount available and the number of oranges that can be purchased.

$$6 : 9 = 12 : x$$

or $6x = 12 \times 9$

or $\dfrac{6x}{6} = \dfrac{9 \times 12}{6}$

or $x = 18$

Hence the required number of oranges = 18

EXERCISE NO. 5.2

1. Find the replacements for the frames in the following proportions :

   (i) $5 : 8 = \square : 6\frac{2}{5}$

   (ii) $8 : \square = 12 : 15$

2. 15 spoons of sugar is required for 6 eggs cake. How many spoons of sugar will be required for 8 eggs cake?

3. 24 kilometers are represented by 3 centimeters on a map. How many centimeters long line-segment will represent an 60 kilometers long road ?

4. The lengths of two iron bars of the same thickness are 6 cms. and 14 cms. respectively. What would be the weight of the second bar if the weight of the first bar is 3 hectograms ?

5. A truck covers a distance of 72 kilometers in 4 hours. How much distance would it cover in 6 hours ?

6. If the price of 12 kilograms of rice is Rs.45, how much rice can be bought for Rs.56. 25.

7. If the price of 12 meters of cloth is Rs.19.50, how much cloth can be bought for Rs.32.50 ?

8. If the price of 10½ meters of 'Lattha' is Rs.28, what would be the price of 16½ meters of 'lattha'?

9. 6 quintals of firewood was bought for Rs.46.50, how much firewood can be bought for Rs.85.25 ?

10. The distance between Lahore and Rawalpindi is 290 kilometers. A train runs 48 kilometers an hour. What time would it take to cover this distance ?

5.4 INVERSE PROPORTION

Suppose $A = \{3,\ 4,\ 6\}$

$B = \{4,\ 3,\ 2\}$

Consider their correspondence as given below

$$\begin{array}{rcccc} A = & \{ 3, & 4, & 6 & \} \\ & \updownarrow & \updownarrow & \updownarrow & \\ B = & \{ 4, & 3 & 2 & \} \end{array}$$

In this correspondence $3 \times 4 = 4 \times 3 = 6 \times 2$ which means that the products of corresponding pairs having mutual correspondence is the same. In such a case we say that the correspondence between A and B is inverse or A and B are inversely proportional.

In the above example

$$\frac{\frac{1}{3}}{4} = \frac{\frac{1}{4}}{3} = \frac{\frac{1}{6}}{2}$$

$\therefore$ Ordered $\left\{\frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right\} \propto$ ordered $\{4, 3, 2\}$

and $$\frac{3}{\frac{1}{4}} = \frac{4}{\frac{1}{3}} = \frac{6}{\frac{1}{2}}$$

$\therefore$ Ordered $\{3, 4, 6\} \propto$ ordered $\left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right\}$

Hence two sets are said to be inversely proportional if there exists a direct proportion between the elements of one set and the multiplicative inverses of the elements (keeping in view the given order) of the other set. If the elements of one set increase in magnitude then the corresponding elements of the other set decrease in magnitude in the same ratio.

Some examples of inverse proportion :

$$(1)\quad \{2,\ 5,\ 6\} \qquad\qquad (2)\quad \{3,\ 2,\ 1\}$$

$$\updownarrow\ \ \updownarrow\ \ \updownarrow \qquad\qquad\qquad \updownarrow\ \ \updownarrow\ \ \updownarrow$$

$$\{15,\ 6,\ 5\} \qquad\qquad\qquad \left\{\frac{1}{3},\ \frac{1}{2},\ 1\right\}$$

Example 1

In a house 14 inmates have food for 30 days. For how many days will the same food be sufficient if 4 of the persons leave the house ?

Solution

If the number of persons decreases the number of days will increase. So it is a problem of inverse proportion.

Suppose the required number of days is x. There is an inverse proportion between the sets.

$$\{14 \text{ persons} \qquad 10 \text{ persons}\}$$

$$\updownarrow \qquad\qquad\qquad \updownarrow$$

$$\{30 \text{ days} \qquad\qquad x \text{ days}\}$$

i.e. there is a direct proportion between

$$\{14 \text{ persons} \qquad 10 \text{ persons}\}$$

and

$$\left\{\frac{1}{30} \text{ days} \qquad \frac{1}{x} \text{ days}\right\}$$

$$\therefore \quad \frac{14}{10} = \frac{\frac{1}{30}}{\frac{1}{x}} \qquad \text{Or} \quad \frac{14}{10} = \frac{1}{30} \times \frac{x}{1} = \frac{x}{30}$$

$$\text{or} \quad \frac{14}{10} = \frac{x}{30} \qquad \therefore x = \frac{14 \times 30}{10}$$

Hence number of days = 42

Alternate Method

Men	Days
14	30
10	x

The smaller the number of men consuming food, the greater will be the period for which the food will last. Therefore, the number of men and days are inversely proportional.

$\therefore \quad 10 : 14 = 30 : x$

or $\quad 10 \times x = 14 \times 30$

or $\quad \dfrac{10x}{10} = \dfrac{14 \times 30}{10}$

or $\quad x = 42$

$\therefore$ Required number of days = 42

EXERCISE NO.5.3

1. Some quantity of milk is sufficient for 120 persons at the rate of ½ litre a person. For how many persons will the same quantity of milk be sufficient if each person is given 3/8 litre of milk ?

2. A store of food is sufficient for 250 soldiers for 21 days. For how many days will the same food be sufficient if 100 more soldiers arrive ?

3. 400 persons had a food stock for 6 days. How many persons should leave so that the same food is sufficient for 8 days ?

4. A train covers a certain distance in 6 hours at the speed of 27 kilometers per hour. In how many hours will it cover the same distance if the speed becomes 24 kilometers ?

5. A train covers the distance between two cities in 4 hours and 12 minutes at the speed of 30,000 meters per hour. What time would it take to cover the same distance if it runs at the speed of 28,000 meters per hour ?

6. 6 men or 12 women complete a work in 10 days. In how many days will 14 men and 12 women complete the same work ?

7. In a fort 350 men had food for 84 days. 30 of them left the fort after 20 days. For how many days will the remaining food suffice ?

8. In a Girl Guides camp, 80 girls had food for 18 days. After 3 days, 40 more girls joined them. For how many days will the remaining food last ?

5.5 COMPOUND PROPORTION

If a problem involves more than one proportions then it is said to involve compound proportion.

See the solved examples below for explanation :

Example 1

30 labourers dig 828 cubic centimeters of earth in 6 hours. How much earth would be dug by 35 labourers in 5 hours ?

Solution

30 labourers dig 828 cubic centimeters of earth in 6 hours.

First of all we shall find how much of earth would 30 labourers dig in 5 hours .

Suppose the earth dug is x cubic c.m.

$$\therefore \quad 6 : 5 = 828 : x$$

$$\text{or} \quad 6x = 828 \times 5$$

$$\therefore \quad x = \frac{828 \times 5}{6}$$

Now we shall find as to how much of earth would 35 labourers dig in 5 hours. Suppose it to be y cubic c.m.

$$30 : 35 = \frac{828 \times 5}{6} : y$$

or $$30y = \frac{35 \times 828 \times 5}{6}$$

or $$y = \frac{35 \times 828 \times 5}{6 \times 30} = 805 \text{ cubic centimeters.}$$

Alternate Method

Since with the increase in the number of labourers the amount of earth dug also increases and same is the case when the time increases, therefore, this problem involves two direct proportions.

Suppose the required quantity of earth is x.

The two proportions can be written in the following form :

$$\begin{matrix} 30 & : & 35 \\ 6 & : & 5 \end{matrix} :: 828 : x$$

or $$30 \times 6 \times x = 828 \times 35 \times 5$$

$$x = \frac{828 \times 35 \times 5}{30 \times 6} = 805 \text{ cubic centimeters.}$$

Example 2

5,500 persons have food which is sufficient for 36 days at the rate of 14 hectograms per person. How many persons should leave so that the same food is sufficient for 44 days at the rate of 15 hectograms per person.

Solution

Since with the decrease in the number of persons, the food will be finished in more days and with the increase in the ration per person the food will be finished in less days, therefore, the problem involves two inverse proportions.

The two proportions can be written in the following compound form :

$$\begin{array}{c}15 \ ; \ 14 \\ 44 \ : \ 36\end{array} :: 5500 : x$$

x is supposed to be the number of persons left.

$$\therefore \quad 15 \times 44 \times x = 14 \times 36 \times 5500$$

$$\text{So} \quad x = \frac{14 \times 36 \times 5500}{15 \times 44} = 4200$$

Hence the rquired number of persons who should leave = 5500 – 4200 = 1300. Since both the proportions are 'inverse', therefore, we write 15 : 14 and 44 : 36 instead of 14 : 15 and 36 : 44 respectively.

Example 3

A contractor undertook the contract of completing a building in 40 days and employed 30 men for the purpose. Only $\frac{1}{3}$ of the work was completed in 16 days. How many more men should he employ to complete the work in fixed time?

Solution

Suppose total number of men required = x

Portion of work already completed = 1/3

Portion of work left = 1 – 1/3 = 2/3

Days left = 40 – 16 = 24

Days	Portion of work	Men
16	1/3	30
24	2/3	x

To execute the work in more days, less men would be required (Inverse proportion)

Greater portion of work can be completed by greater number of men .(Direct proportion)

$$\therefore \quad \begin{array}{c}24 \ : \ 16 \\ \frac{1}{3} \ : \ \frac{2}{3}\end{array} :: 30 : x$$

$$24 \times \frac{1}{3} \times x = 16 \times \frac{2}{3} \times 30$$

$$x = 16 \times \frac{2}{3} \times 30 \times \frac{1}{24} \times \frac{3}{1} = 40$$

Therefore additional number of men to be employed

$$40 - 30 = 10 \text{ men}$$

EXERCISE NO.5.4

1. The fare for carrying a luggage weighing 70 kilograms for 150 kilometers is Rs.5. What would be the fare for carrying a luggage weighing 105 kilograms for 300 kilometers ?

2. The fare for carrying a luggage weighing 10 quintals for 16 kilometers is Rs.10. What would be the fare for carrying a luggage weighing 7 quintals for 20 kilometers ?

3. 8 masons can build a 10 meters long wall in 22 days. How many masons would be required to build a 165 meters long wall in 6 days.

4. 56 women can spin 42 kilograms of thread by working 10 hours a day. How much thread will 50 women spin if they work 6 hours daily ?

5. A family's expenditure on rice is Rs.12 for 16 days if the rate of rice is Rs.500 per quintal. What would be the expenditure for 5 days if the rate is reduced to Rs.300 per quintal ?

6. A contractor undertook a contract to build a road in 20 days and employed 30 men to work. Only 1/3 of the work was completed after 8 days. How many more men should be employed so that the work is completed within the fixed time ?

7. In a fort 800 men had food for 30 days. 400 more men joined them after 10 days. For how many days with the remaining food be sufficient if the amount of ration per person is reduced to 2/3 of the earlier quantity ?

8. 1,200 persons had food for 40 days. How many persons should leave so that the same food is sufficient for 48 days when the amount of ration per person is increased to 1¼ times of the original quantity ?

9. 30 labourers were employed to complete a work in 8 days. But only 1/3 of the work could be completed after 3 days. How many more labourers should be employed to complete the work on time ?

10. The price of a 6 meters long and 3 meters wide carpet is Rs.2700. What would be the price of a carpet 8 meters long and 1½ meters wide ?

5.6 CONTINUED RATIOS

Sometimes we have to compare more than two quantities. For example Iftikhar, Ansar and Rashid jointly started a business by investing some money. The ratio between Iftikhar's and Ansar's capitals is 2 : 3 and the ratio between Ansar's and Rashid's capitals is 3 : 4. What is the ratio between the capitals of Iftikhar and Rashid ?

Iftikhar	:	Ansar	:	Rashid
2	:	3		
		3	:	4
2	:	3	:	4

The ratio between the capitals of Iftikhar and Rashid is 2 : 4. Comparing in such a way is called comparing with the help of continued ratio . This is a precise way of comparing quantities.

Example 1

Divide 860 rupees among Ayub, Azam and Akbar in such a way that if Ayub gets 5 rupees then Azam gets 4 rupees and if Azam gets 6 rupees then Akbar gets 8 rupees.

Solution

There are two ratios in the above question, namely :

Ayub's share : Azam's share = 5 : 4

Azam's share : Akbar's share = 6 : 8

According to first ratio, Azam gets 4 rupees and according to second ratio he gets 6 rupees. In the two ratios Azam's share is common. In order to make Azam's share the same in both the ratios, we take L.C.M. of 4 and 6 which is 12. To make Azam's share 12 in the first ratio,we have to multiply 4 by 3 and to make his share 12 in the second ratio,we have to multiply 6 by 2.

∴ To keep the two ratios unchanged, we have to multiply the first and second ratio by 3 and 2 respectively.

	Ayub's share	:	Azam's share
=	5	:	4
=	3 x 5	:	3 x 4 (multiplying the terms by 3)
=	15	:	12

	Azam's share	:	Akbar's share
=	6	:	8
=	2 x 6	:	2 x 8 (Multiplying the terms by 2)
=	12	:	16

Ayub's share : Azam's share : Akbar's share

15 : 12 : 16

The process of finding continued ratio can be made shorter in the following manner.

Ayub		Azam		Akbar
5	:	4		
		6	:	8
5 x 6	:	6 x 4	:	4 x 8
30	:	24	:	32
15	:	12	:	16

Or

Ayub		Azam		Akbar
5	:	4		
		6		8
5 x 6	:	6 x 4	:	4 x 8
30	:	24	:	32
15	:	12	:	16

The method on the right is known as the 'Method of Inverted N' as the arrows make the symbol "\/\".

Sum of ratios = 15 + 12 + 16 = 43

Amount to be divided = 860 rupees

$$\text{Ayub's share} = \frac{15}{43} \times 860 = \text{Rs.}300$$

$$\text{Azam's share} = \frac{12}{43} \times 860 = \text{Rs.}240$$

$$\text{Akbar's share} = \frac{16}{43} \times 860 = \text{Rs.}320$$

Note : If there are four persons and three pairs of ratios, then first find the continued ratio between the three and then with the fourth.

Example 2

Divide Rs.150 among A, B, C and D in such a way that A's share : B's share = 2 : 3, B's share : C's share = 1 : 2 and C's share D's share = 3 : 2.

Solution

$$\therefore \begin{array}{ccccc} \text{A's share} & : & \text{B's share} & : & \text{C's share} \\ 2 & : & 3 & & \\ & & 3 & : & 6 \\ \hline 2 & : & 3 & : & 6 \end{array}$$

Or Briefly

$$\begin{array}{ccccc} \text{A's Share} & : & \text{B's Share} & : & \text{C's Share} \\ 2 & : & \uparrow 3 & & \\ & & \rightarrow 1 & : & 2 \\ \hline 2 & : & 3 & : & 6 \end{array}$$

C's share : D's share = 3 : 2

C is the common link between A, B, C and C, D. Therefore, we have to equalise it in both cases to get the continued ratio between A, B, C, D.

3 : 2 is equivalent to 6 : 4

∴	A's share	:	B's share	:	C's share	:	D's share
	2	:	3	:	6		
					6	:	4
	2	:	3	:	6	:	4

Sum of ratios $= 2 + 3 + 6 + 4 = 15$

$$\text{Hence A's share} = \frac{150 \times 2}{15} = 20 \text{ rupees}$$

$$\text{B's share} = \frac{150 \times 3}{15} = 30 \text{ rupees}$$

$$\text{C's share} = \frac{150 \times 6}{15} = 60 \text{ rupees}$$

$$\text{D's share} = \frac{150 \times 4}{15} = 40 \text{ rupees}$$

5.7 PROPORTIONAL DIVISION

A division of some quantity into two or more than two shares in a certain ratio is called "Proportional Division."

Some solved examples are given below to explain the method

Example 1

Divide 450 rupees among Ahmad, Bashir and Jamil in the ratio of 2, 3 and 4 respectively.

Solution

	Ahmad's share	;	Bashir's share	:	Jamil's share
=	2	:	3	:	4

Sum of ratios $= 2 + 3 + 4 = 9$

$$\text{Ahmad's share} = \frac{450 \times 2}{9} = 100 \text{ rupees}$$

$$\text{Bashir's share} = \frac{450 \times 3}{9} = 150 \text{ rupees}$$

Jamil's share $= \dfrac{450 \times 4}{9} = 200$ rupees

Example 2

Divide 4,650 rupees in the ratio of $\dfrac{1}{2} : \dfrac{1}{3} : \dfrac{1}{5}$

Solution

$$\frac{1}{2} : \frac{1}{3} : \frac{1}{5}$$

$$= \frac{1}{2} \times 30 : \frac{1}{3} \times 30 : \frac{1}{5} \times 30$$ (multiplying each term by 30, i.e. L. C. M. of 2, 3, 5)

$$= 15 : 10 : 6$$

Sum of ratios $= 15 + 10 + 6 = 31$

Amount to be divided $= 4650$ rupees

First share $= \dfrac{15 \times 4650}{31} = 2250$ rupees

Second share $= \dfrac{10 \times 4650}{31} = 1500$ rupees

Third share $= \dfrac{6 \times 4650}{31} = 900$ rupees

EXERCISE NO.5.5

(1) Divide 150 rupees among Saleem, Kaleem and Raheem in the ratio of 4 : 5 : 6.

(2) Divide Rs.32.50 among A, B, C in the ratio

$$\frac{3}{4} : \frac{5}{6} : \frac{7}{12}$$

(3) Divide a profit of 976 rupees among A, B, C and D in the ratio $1 : \dfrac{3}{4} : \dfrac{3}{5} : \dfrac{7}{10}$

(4) Zaid, Umar and Bakar jointly started a business. After a year they gained a profit of 1,230 rupees. Divide the amount of profit among them in the ratio of 14, 15 and 12 respectively.

(5) Three fishermen caught 690 fish. If the ratio between their fish is $\frac{2}{3}$, $\frac{4}{5}$ and $\frac{5}{6}$, how many fish did each of them catch?

(6) The ratio of water, milk and sugar in a cup of tea is $1 : \frac{1}{8} : \frac{3}{16}$. Find the weight of water, milk and sugar in 1 kilogram 50 grams of tea.

(7) Divide 1,036 rupees among Zafar, Mazhar and Anwar so that the ratio between the shares of Zafar and Mazhar is 3 and 4 and the ratio between the shares of Mazhar and Anwar is 3 : 4.

(8) Divide 5428 rupees among A, B and C, so that the ratio between the shares of A and B is 3 and 4 ; and the ratio between the shares of B and C is 5 and 6.

(9) Tahir, Hamid and Iqbal have to divide among themselves a profit of Rs.443.75 so that if Hameed gets 6 rupees, Tahir gets 5 rupees; and Iqbal gets 9 rupees if Hameed gets 8 rupees.

(10) Divide 1,550 rupees among A, B, C and D so that the ratio between the shares of A and B is 2 : 3 ; the ratio between the shares of B and C is 2 : 3 and the ratio between the shares of C and D is 3 : 4.

5.8 MISCELLANEOUS PROBLEMS

Example 1

A, B and C have a total amount of Rs.390. If A had 5 rupees, B had 7 rupees and C had 8 rupees less than what each one of them actually has, the ratio between their remaining sums would have been 9, 13 and 15 respectively. How much money does each of them possess ?

Solution

Total amount = Rs.390

Rupees 5, 7 and 8 have to be taken away from the shares of A, B and C respectively as the given ratio holds among only the remainders of the shares. A total of Rs.20, (5 + 7 + 8), is thus subtracted from Rs.390. This leaves us with a balance of Rs.370.

Ratio in the remainders of the shares = 9 : 13 : 15

Sum of ratios = 9 + 13 + 15 = 37

A's share out of Rs.370 = $\frac{9}{37} \times 370$ = Rs. 90

A's actual amount = 90 + 5 = 95 Rs.

B's share out of Rs. 370 = $\frac{13}{37} \times 370$ = Rs.130

B's actual amount = 130 + 7 = 137 Rs.

C's share out of Rs.370 = $\frac{15}{37} \times 370$ = Rs.150

C's actual amount = 150 + 8 = 158 Rs.

Example 2

Divide 290 rupees in three shares so that $\frac{2}{3}$ of first, $\frac{3}{4}$ of second and $\frac{1}{2}$ of third share are all equal.

Solution

Suppose $\frac{2}{3}$ of the first share = $\frac{3}{4}$ of the second share = $\frac{1}{2}$ of the third share = Rs.1

First share = $1 \times \frac{3}{2}$ = Rs $\frac{3}{2}$

Second share = $1 \times \frac{4}{3}$ = Rs $\frac{4}{3}$

Third share = $1 \times \frac{2}{1}$ = Rs.2

Hence Ist share : 2nd share : 3rd share

$$\frac{3}{2} \quad : \quad \frac{4}{3} \quad : \quad 2$$

or 1st share : 2nd share : 3rd share = 9 : 8 : 12 (Multiplying the ratio's by 6, the L.C.M. of denominators 2 and 3)

Sum of ratios = 9 + 8 + 12 = 29

Total amount of money = 290 rupees.

First share = $\frac{290 \times 9}{29}$ = 90 rupees

Second share = $\frac{290 \times 8}{29}$ = 80 rupees

Third share = $\frac{290 \times 12}{29}$ = 120 rupees

Example 3

Divide Rs.12540 among A, B and C so that A gets $\frac{3}{7}$ of the total of the shares of B and C ; and B gets $\frac{2}{9}$ of the total of the shares of C and A.

Solution

A gets $\frac{3}{7}$ of the total shares of B and C . If the total of the shares of B and C is 7 rupees, A gets $\frac{3}{7}$ of 7 = 3 rupees.

Thus the total of the shares of A, B and C = 7 + 3 = 10 rupees i.e. A gets $\frac{3}{10}$ of the total amount.

$\therefore$ A's share $= \frac{3}{10} \times 12540 = 3762$ rupees

Similarly B gets $\frac{2}{9}$ of the total shares of C and A. i.e. if the total of the shares of C and A is 9, B's share $= \frac{2}{11}$ of the total amount.

Hence B's share $= \frac{2}{11} \times 12540 = 2280$ rupees

A's share = 3762 rupees

B's share = 2280 rupees

Total of the shares of A and B = 3762 + 2280 = 6042 rupees.

$\therefore$ C's share = 12540 − 6042 = 6498 rupees

Example 4

Divide 315 rupees among 5 men 6 women and 10 boys so that if a boy gets 1 rupee, a woman gets 1¼ rupees and a man get 1¾ rupees. Find share of each one of them.

Solution

(i) If one man gets 1¾ rupees,

5 men will get $5 \times 1\frac{3}{4} = 5 \times \frac{7}{4} = \frac{35}{4}$ rupees

(ii) One woman gets 1¼ rupees,

6 women will get $6 \times 1\frac{1}{4} = 6 \times \frac{5}{4} = \frac{30}{4}$ rupees

(iii) One boy gets 1 rupees

10 boys will get 10 x 1 = 10 rupees

Continued ratio among the share of

5 men , 6 women and 10 boys

$= \frac{35}{4} : \frac{30}{4} : 10$

$= 35 : 30 : 40$

Sum of ratios $= 35 + 30 + 40 = 105$

Total amount $= 315$ rupees

Share of 5 men $= \frac{35}{105} \times 315 = 105$ rupees

Share of one man $= \frac{105}{5} = 21$ rupees

Share of 6 women $= \frac{30}{105} \times 315 = 90$ rupees

Share of one woman $= \frac{90}{6} = 15$ rupees

Share of 10 boys $= \frac{40}{105} \times 315 = 120$ rupees

Share of one boy $= \frac{120}{10} = 12$ rupees

EXERCISE NO.5.6

1. Three men have Rs.630 in all. If each of them gives away 10 rupees in charity, the ratio between their remaining shares is 3 : 4 : 5. Find their actual shares.

2. Amjad, Asghar and Asad get 126 rupees in all. If Amjad had got 5 rupees more, Asghar had got 9 rupees more and Asad had got 10 rupees more, the ratio between their shares would have been 4, 5 and 6 respectively. Find shares of each one of them.

3. Munir, Bashir and Nazeer got prizes of 192 rupees. Had they won prizes of 11, 15 and 13 rupees more respectively, the ratio between their shares would have been 7, 6 and 8 respectively. How much did each one of them get ?

4. Divide 81 into 3 shares so that half of first, one-third of second and one-fourth of third are all equal.

5. Divide 195 rupees among Ameena, Nasreen and Yasmeen so that twice of Ameena's share, three times of Nasreen's share and three-fourths of Yasmeen's share are all equal.

6. Divide 1250 rupees among A, B, and C so that A gets $\frac{2}{3}$ of the total shares of B and C, and B gets $\frac{9}{16}$ of the total of the shares of A and C.

7. Divide a profit of 715 rupees among Hameed, Waheed and Saeed so that Hameed's share is 4/7 of the others' shares and Waheed's share is 5/8 of the others' shares.

8. Divide 112 rupees among A, B and C so that A's share is 3/4 of the total shares of B and C and B's share is 3/5 of the total of the shares of C and A.

9. Divide 320 rupees among 5 men and 6 women so that each man gets twice as much as a woman gets.

10. Dive Rs.26.25 among A, B and C so that B gets 5/6 of A's share and C gets twice the share of B.

11. An amount of Rs.670 is to be divided among 4 boys 5 men and 8 women in such a way that if a man gets Rs.2.50 a boy gets Rs.1.25 and a woman receives Rs.2 . Find separately the share of a boy, a man and a woman.

12. There are two officers, three peons and seven clerks in an office. Their total monthly salary of Rs.12500 is disbursed in such a way that if a peon gets Rs.1 , a clerk gets Rs.2 and an officer gets Rs.4. Find the monthly salary of each employee.

5.9 INHERITANCE

"Tarka" (ترکہ) : The property which someone (man or woman) leaves behind him on his death is called "Tarka". This Tarka is divided among his kins and relatives. Before dividing "Tarka" it is necessary that any loan due on the deceased should be paid and the payment should be made according to his will. The dower of wife is also considered a loan. The right of inheritance is established on the death of someone and not in his life. Some examples are given below to explain the method of distributing 'Tarka'.

Example 1

A person on his death, left a property of 14,400 rupees which was distributed among his wife, mother, father, a son and a daughter. Widow got $\frac{1}{8}$, mother $\frac{1}{6}$ and father $\frac{1}{6}$ of the property. The remaining property was distributed between his son and daughter in the ratio of 2 : 1. Find the shares of each one of them.

Solution

Sum of shares of widow, mother and father

$$= \frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}$$

Remaining property $= 1 - \frac{11}{24} = \frac{13}{24}$

Son's share : daughter's share = 2 : 1

Sum of ratios = 2 + 1 = 3

Sons's share $= \frac{2}{3} \times \frac{13}{24} = \frac{13}{36}$

Daughter's share $= \frac{1}{3} \times \frac{13}{24} = \frac{13}{72}$

The ratio of the shares of heirs

Widow, mother, Father, a son, a daughter

$$= \frac{1}{8} : \frac{1}{6} : \frac{1}{6} : \frac{13}{36} : \frac{13}{72}$$

$$= 9 : 12 : 12 : 26 : 13$$

(multiplying by 72)

Sum of ratios $= 9 + 12 + 12 + 26 + 13 = 72$

Widow's share $= \frac{9}{72} \times 14400 = 1800$ rupees

Mother's share $= \frac{12}{72} \times 14400 = 2400$ rupees

Father's share $= \frac{12}{72} \times 14400 = 2400$ rupees

Son's share $= \frac{26}{72} \times 14400 = 5200$ rupees

Daughter's share $= \frac{13}{72} \times 14400 = 2600$ rupees

Example 2

Mst. Razia Begum died leaving behind her a property of 8,025 rupees which was distribtued among her heirs, her husband, her mother and two daughters. The husband got $\frac{1}{4}$, mother $\frac{1}{6}$, and daughters $\frac{2}{3}$ of the property. Find the share of each one when 225 rupees were spent on burial.

Solution

The remaining property after subtracting expenses on burial $= 8025 - 225 = 7800$ rupees.

The ratio of the shares of husband, mother and two daughters

$$= \frac{1}{4} : \frac{1}{6} : \frac{2}{3} = 3 : 2 : 8$$

Sum of ratios $= 3 + 2 + 8 = 13$

Husband's share $= \frac{3}{13} \times 7800 = 1800$ rupees

Mother's share = $\frac{2}{13}$ x 7800 = 1200 rupees

2 daughters share = $\frac{8}{13}$ x 7800 = 4800 rupees

Share of each daughter = 2400 rupees

Example 3

After paying a debt of 3,000 rupees out of a 'Tarka' of 15,000 rupees, the remaining amount is to be distributed among widow, 4 daughters and 2 sons so that the widow gets $\frac{1}{8}$ of the property and out of the remaining property every son gets twice as much as every daughter.

Solution :

Remaining Tarka after payment of debt
= Rs. 15000 – Rs.3000 = Rs.12000

Widow's share = $\frac{1}{8}$ of the balance after payment of of the debt = $\frac{1}{8}$ x 12000 = Rs.1500

Value of remaining property = Rs.12000 – Rs.1500 = Rs. 10500

The amount is to be divided among 4 daughters and 2 sons.

If share of one daughter = Rs.1

Share of each son = Rs.2

Share of 4 daughters ; Share of 2 sons
= 1 x 4 : 2 x 2
= 4 : 4
= 1 : 1 (Dividing by 4)

Sum of ratios = 1 + 1 = 2

Share of 4 daughters = $\frac{1}{2}$ x 10500 = Rs.5250

Share of 1 daughter $= \frac{5250}{4} =$ Rs.1312.50

Share of 2 sons $= \frac{1}{2} \times 10500 =$ Rs.5250

Share of 1 son $= \frac{5250}{2} =$ Rs.2625

EXERCISE NO.5.7

1. A person died leaving behind him a 'Tarka' worth Rs.24,000. Distribute the 'Tarka' among the deceased's mother widow and a son so that the mother gets $\frac{1}{6}$, widow $\frac{1}{8}$ of the property and the remaining goes to the son while the payment of a debt of 4,000 rupees and the dower amount of 5,000 rupees is due from him.

2. A woman died. Her heirs were her husband, two daughters and a brother. A 'Tarka' worth rupees 2,700 was distributed among her heirs in the ratio $\frac{1}{4} : \frac{2}{3} : \frac{1}{12}$ respectively. Find the share of each.

3. Wife of Zaid died leaving behind her two daughters and a son. Zaid got $\frac{1}{4}$ of 'Tarka' of 18,000 rupees. The remaining sum was deposited in the account of her children so that each daughter got half of what a son got. Find the share of her son and each daughter.

4. A person died leaving behind him a 'Tarka' worth Rs.30,000. Distribute the 'Tarka' among 4 sons and 3 daughters so that each daughter gets half of what a son gets. Find the share of each when a debt of Rs.8,000 was also paid out of the 'Tarka'.

5. Munshi Kareem died leaving behind him a 'Tarka' worth Rs:56,000. If a debt of 2,000 rupees was due from him, find the share of each one of the heirs according to the following entitlement.

Widow's share $= \frac{1}{8}$ mother's share = father's share $= \frac{1}{6}$

share of two daughters $= \frac{2}{3}$

6. A person died leaving behind him 4 daughters, 6 sons and a widow. Wife got $\frac{1}{8}$ of the property and each son got twice of what each daughter got. What will be the share of each one out of a 'Tarka' of 20,000 rupees when a debt of 4,000 rupees is also due on the deceased.

7. On the death of a person his heirs were 3 daughters, his mother and his father. A debt of 15,000 rupees was due on him. Find the shares of every heir out of a 'Tarka' worth Rs.51,000 so that his father and mother each get $\frac{1}{6}$ of the property and each daughter gets $\frac{2}{9}$ of it.

8. Bakar died childless while his parents had also died. A debt of 4,000 rupees was due on him and 5,000 rupees were given to his friend Abid. According to his will, $\frac{1}{3}$ of the remaining amount went to a school and the remaining was distributed among his two brothers and a sister so that a brother got twice of what a sister got. Find the share of each brother and sister while the value of his 'Tarka' was 42,000 rupees.

9. The 'Tarka' of a person was distributed according to his will so that 8,000 rupees were paid as debt, and $\frac{1}{6}$ of the remaining property was given in chairty. The balance was distributed among 3 sons and a daughter so that each son got twice of what a daughter received . Find the share of each if the value of his 'Tarka' was 50,000 rupees.

5.10 PARTNERSHIP

Sometime two or more persons combine their resources and start a joint business. They share its risks and profits. The association of persons joined as partners in business is called 'partnership' In partnership the business is managed suitably and concerned problems are solved in a better way. The money invested in a business is called 'capital'. The partnership in which capital of every partner remains invested for the same period of time is called 'simple partnership'. In such a case, the profit or loss is distributed among the partners in the ratio of their investments.

Example 1

A, B and C started a business. A invested 6,000 rupees, B invested 9,000 rupees and C invested 12,000 rupees. At the end of a year they gained 5,400 rupees as profit. Find the share of each in the profit.

Solution

Since the capital of each one of them remained in business for the whole year, the profit is to be distributed among them according to the ratios of their capitals.

A's capital : B's capital : C's capital

6000 : 9000 : 12000 = 6 : 9 : 12

= 2 : 3 : 4

Sum of ratios = 2 + 3 + 4 = 9

Total profit = 5400 rupees.

A's share in profit = $\frac{2 \times 5400}{9}$ = 1200 rupees

B's share in profit = $\frac{3 \times 5400}{9}$ = 1800 rupees

C' share in profit = $\frac{4 \times 5400}{9}$ = 2400 rupees

Example 2

A, B and C started a joint business with a total investment of Rs.10,500 and gained a profit of Rs.472.50. A invested 1,000 rupees more than B and 2,000 rupees more than C. Find the share of each partner in the profit.

Solution

We have to first find the investment of each partner out of Rs.10,500 for distributing the profit.

'A invests Rs.1000 more than B and Rs.2000 more than C means that

(i) A's capital is Rs.2000 more than C.

(ii) B's capital is Rs.1000 more than C.

If we set apart Rs.2,000 + Rs.1,000 = Rs.3,000 from the total investment;

Divide the remaining amount of Rs.7500 (Rs.10500 – Rs.3000) equally among them and then;

Add Rs.2,000 and Rs.1,000 to the investments of A and B for whom they have been set apart, we will get the investment of each partner separately.

Share of each partner out of Rs.7500 = $\frac{7500}{3}$ = Rs.2500

∴ C's chare = Rs.2500

B's share = Rs.2500 + Rs.1000 = Rs.3500

A's share = Rs.2500 + Rs.2000 = Rs.4500

A's capital : B's capital : C's capital

4500 : 3500 : 2500

45 : 35 : 25

9 : 7 : 5

Sum of ratios = 9 + 7 + 5 = 21

A's share in profit = $\frac{9}{21} \times 472.50$ =Rs.202.50

B's share in profit = $\frac{7}{21} \times 472.50$ = Rs.157.50

C's share in profit = $\frac{5}{21} \times 472.50$ = Rs.112.50

EXERCISE NO.5.8

1. Zaid, Umar and Bakar started a joint business and invested 840 rupees, 900 rupees and 720 rupees respectively. They gained a profit of 1,230 rupees after a year . Find each one's share in the profit.

2. A invested 6,300 rupees, B invested 4,200 rupees and C invested 7,500 rupees in a joint business and gained a profit of 720 rupees. Find each one's share in the profit.

3. Kareem invested 3,600 rupees, Rahim invested 9,900 rupees and Saleem invested 7,200 rupees in a joint business and gained a profit of Rs.885.50 . Find each one's share in the profit.

4. In a business 1,240 rupees were gained as profit. Find the share of each partner if they invested 960 rupees, 864 rupees and 1,152 rupees respectively.

5 A, B, and C invested 3,000 rupees 3,250 rupees and 3,500 in a joint business. After a year they gained a profit of 7,240 rupees. Find the share of each one if A got 1,000 rupees out of profit money for acting as manager.

6. A, B, C and D invested rupees 2,000, 1875, 875 and 1,625 respecitively in a joint business and gained a profit of 3,240 rupees after a year. Find the share of each one if C has to get 100 rupees per month for acting as manager

7. A, B and C started a business. They invested 4,600 rupees and gained a profit of 690 rupees. Find each one's share in the profit if A invested 400 rupees more than those of B and B invested 300 rupees more than those of C.

Note : A partner who manages the business is called an 'active partner'. The partner who only invests money and takes no part in running the business is called a 'sleeping partner'. The profit is distributed among the partners according to the ratios in their capitals after paying the salary to the active parnter.

5.11 COMPOUND PARTNERSHIP

In partnership it usually happens that a person starts business Another joins him after sometime and then a third one joins them. In this way the capitals of different partners remain in the business for different periods. Such a partnership is called Compound Partnership. It is obvious that in such a situation the profit cannot be distributed according to the ratios in the capitals of the partners because the period of investment differs from partner to partner.

When capitals are invested for different periods, it is necessary to bring them to the same level for comparison. The statement that Aslam invests Rs.4000 for 6 months and Majid invests Rs.3500 for 8 months furnishes us no basis for direct comparison.

If, however, we ponder over the question as to how much capital should Aslam invest to get the same profit in one month which he otherwise gets by investing Rs.4000 for 6 months, the answer is Rs.4000 x 6 = 24000 n other words the investment of Rs.24000 (4000 x 6) would earn him the same profit in one month which Rs.4000 earn him in 6 months.

Similarly Rs.3500 of Majid invested for 8 months would bring as much profit as Rs.3500 x 8 invested for one month. This establishes the following basis for comparison.

When different capitals are invested for different periods, find the equivalent investments for 1 month by multiplying the capital of each partner with the number of months for which the capital actually remains in business.

The following examples explain the method.

Example 1

A, B and C started a joint business. A invested 9,000 rupees for 6 months, B invested 6,000 rupees for 4 months and C invested 7,500 rupees for 2 months. Find the share of each one in a profit of 1,085 rupees.

Solution

A invested 9,000 rupees for 6 months, therefore, he invested 9,000 x 6 or 54,000 rupees for one month, B invested 6,000 rupees for 4 months, therefore, he invested 6,000 x 4 or 24,000 rupees for one month, and C invested 7,500 rupees for 2 months, therefore, he invested 7,500 x 2 or 15,000 rupees for one month.

Ratios of their investment

=54000 : 24000 : 15000

= 54 : 24 : 15

= 18 : 8 : 5

Sum of ratios = 18 + 8 + 5 = 31

Profit = 1085 rupees

A's share in profit = $\frac{18 \times 1085}{31}$ = 630 rupees

B's share in profit = $\frac{8 \times 1085}{31}$ = 280 rupees

C's share in profit = $\frac{5 \times 1085}{31}$ =175 rupees

Example 2

A and B started a joint business. They invested 1,500 rupees and 1,200 rupees respectively. A took out onethird of his capital after 4 months and B took out onefourth of his capital after 6 months. At this time C joined them by investing 1,800 rupees. Find the share of each one's profit if they gained a profit of 374 rupees after a year.

Solution

A's capital for 4 months = 1500 rupees

A's capital for 1 month = 1500 × 4 = 6000 rupees

A's capital for 8 months = 1500 − $\frac{1}{3}$ of 1500 = 1000 ₨.

A's captial for 1 month = 1000 × 8 = 8000 rupees

A's total investment = 6000 + 8000 = 14000 rupees

B's capital for 6 months = 1200 rupees

B's capital for 1 month = 1200 × 6 = 7200

B's capital for next 6 months = 1200 − $\frac{1}{4}$ of 1200 = 900 rupees

B's capital for 1 month = 900 × 6 = 5400 rupees

B's total investment = 7200 + 5400 = 12600 rupees

C's capital for 6 months = 1800 (∵ he joined when B took out $\frac{1}{4}$ of his capital

C's capital for 1 month = 1800 x 6 = 10800 rupees

Ratios in capitals = 14000 : 12600 : 10800

= 140 : 126 : 108

= 70 : 63 : 54

Sum of ratios = 70 + 63 + 54 = 187

Profit = 374 rupees

A's share in prof it = $\frac{70 \times 374}{187}$ = 140 rupees

B's share in profit = $\frac{63 \times 374}{187}$ = 126 rupees

C's share in profit = $\frac{54 \times 374}{187}$ = 108 rupees

Example 3

A and B started a joint business. The ratios in their capitals was 5 : 6 but their shares in the profit were equal. For what period did B's capital remain in the business, if A's capital remained in the business for 18 months.

Solution

Suppose A's capital = 5 rupees.

A's capital worth rupees 5 for 18 months = 5 x 18 = 90 rupees for 1 month

Since A's share in profit = B's share in profit, therefore, B's capital for one month = 90 rupees.

or B's capital x 6 = 90

∴ B's share remained in the business for

90 ÷ 6 = 15 months.

EXERCISE NO.5.9

1. Rehman started a business with 2,400 rupees in the beginning of a year. Kareem became a partner by investing 7,200 rupees in the middle of the year. They gained a profit of 160 rupees at the end of the year. Find each one's share in the profit.

2. A, B and C started a joint business. A invested 500 rupees for 9 months, B invested 700 rupees for one year and C invested 1,200 rupees for 5 months. They suffered a lost of 945 rupees at the end of the year. Find each one's share in the loss.

3. Three partners invested 2,000 rupees , 3,500 rupees and 4,750 rupees respectively in a joint business. Their capitals remained in the business for 6 months, 3 months and 2 months respectively. What would be the share of each in a profit of 4,000 rupees ?

4. A, B and C jointly rented a pasture for 544 rupees. A grazed his 16 cattle for 6 weeks, B grazed his 22 cattle for 4 weeks and C grazed his 30 cattle for 12 weeks. How much of rent would each pay ?

5. A invested 750 rupees in a business. After 4 months he invested 250 rupees more and at that time B joined A by investing 1,500 rupees. Aftter 3 months B took 500 rupees back and at this time C joined them by investing 5,000 rupees. After a year they gained a profit of 4,264 rupees. Out of this profit B was paid 52 rupees per month for managing the business. Find the share of each in the profit.

7. A, B and C invested in a shop in the ratio 1/2 : 1/3 : 1/4. A took half of his capital out after 4 months 8 months after that they gained a profit of 2,024 rupees. Find the share of each in the profit.

8. A started a business with an investment of 1,200 rupees. After some time B also joined him by investing 1,600 rupees. At the end of a year they divided the profit equally. When did B join?

9. A started a business with a capital of 4,500 rupees. B joined him afterwards by investing 5,400 rupees. If at the end of a year the profit was distributed in the ratio of 2 : 1 , when did B join ?

CHAPTER 6

PERCENTAGE

6.1 PERCENT

You know that 6 parts out of 100 of a quantity is 6 percent of the quantity. Similarly 15 out of one hundred parts means 15 percent. "Percent is denoted by the symbol "%". The use of percent in practical life is explained below with the help of examples :

You know that if 1/4 of a number is 5 then that number is

$$5 \div \frac{1}{4} = \frac{5 \times 4}{1} = 20$$

Similarly if 2/3 of a number is 4, then that number is

$$4 \div \frac{2}{3} = 4 \times \frac{3}{2} = 6$$

If 15 percent of a number is 45, then that number is

$$45 \div \frac{15}{100} = \frac{45 \times 100}{15} = 300$$

If 16 rupees are spent out of 20 rupees, what is the percentage of expenditure ?

Expenditure of 16 out of 20 means $\frac{16}{20}$

Expenditure out of 100 = $\frac{16}{20} \times 100$ = 80 percent

Example 1

The weight of one cubic centimeter of wood is 8 decigrams. (a) what percent is wood's weight of water's weight of the same volume ? (b) what percent is water's weight of the wood's weight of an equal volume ?

(The weight of one cubic centimeter of water is taken as 1 gram).

Solution

(a) Weight of one cubic centimeter of wood

$$= 8 \text{ decigrams} = \frac{8}{10} \text{ grams.}$$

Weight of one cubic centimeter of water = 1 gram

Weight of one cubic centimeter of wood is 8/10 of the weight of water.

$$\text{In percentage} = \frac{8}{10} \times 100 = 80 \text{ percent}$$

(b) Weight of one cubic centimeter of water

$$= 1 \div \frac{8}{10} = \frac{10}{8} \text{ of the weight of wood.}$$

$$\text{In percentage} = \frac{10}{8} \times 100 = 125 \text{ percent}$$

Example 2

35 percent of a sum is 14 rupees. Find that sum

Solution

35 percent of the sum = 14 rupees

$$\therefore \quad \text{Sum} = 14 \div \frac{35}{100} = \frac{14 \times 100}{35} = 40 \text{ rupees}$$

Example 3

A landlord got 40 percent of a crop. What is the total value of the crop if the landlord got 640 rupees ?

Solution

Value of the 40 percent of the crop = 640 rupees

Total value of the crop

$$= 640 \div \frac{40}{100} = \frac{640 \times 100}{40} = 1600 \text{ rupees}$$

EXERCISE NO.6.1

1. Shahzad got 40 rupees as pocket money. He spent 24 rupees. What percentage of pocket money did he spend ?

2. The population of a village was 7,740 in 1961. In 1971 the population rose to 8901. Find percent increase in population.

3. Estimated expenditure of a house was 15,000 rupees but the actual expenditure exceeded it by Rs.3600 . Find the percent increase in expenditure.

4. Income of a person is 750 rupees and expenditure is 630 rupees per month. Find his saving percentage.

5. Of what number is 9 (i) 20 percent, (ii) 15 percent ?

6. 7½ percent of a number is 30. Find the number.

7. A person bought some oranges 20 percent were found to be rotten and the remaining number was 120 . Find the total number of oranges.

8. Shahid did 24 experiments and said that he had completed 75 percent experiments. Find the total number of experiments.

9. On a sale, the price of a pair of shoes was Rs.72 which is 60 percent of the original price. Find its original price.

10. A person paid 2,750 rupees back to the bank. In this way he paid 55 percent of the loan. What was the total amount of loan ?

6.2 ZAKAT

Islam has laid down fundamental principles for all aspects of life. In order to establish equity and fraternity in social and economic life, the rich have been exhorted to look after the economic needs of the poor Muslim brothers. Accordingly in Islam, the rich are duty bound to deposit every year 2½%of their saving in a fund meant for meeting the needs of the poor. This amount is called "Zakat". It is the religious duty of every Muslim who saves in a year 6 hectograms of silver or 8.7 decagrams of gold or any amount equal to the value of any of these metals to give 2½ % of that amount as "Zakat".

Example

A person saved 20,000 rupees in a year. His wife had ornaments of 15,000 rupees. How much Zakat is due from him ?

Solution

Amount on which Zakat is due

$$= 20000 + 15000 = 35000 \text{ rupees}$$

Zakat at the rate of 2½ %

$$= 2.5 \times \frac{35000}{100} = \frac{5 \times 35000}{2 \times 100} = 875 \text{ rupees}$$

6.3 TAX

The government levies taxes on public for the completion of useful projects. New hospitals are established to provide treatment to thousands of people. New educational institutions are opened to educate increasing number of students. Roads, bridges, dams and barrages are constructed. Improvement of agriculture, railways and telecommunications, setting up of new industries, maintenance of judiciary, army, and other departments require funds. The state collects the funds through different types of taxes.

Taxes are Federal, Provincial and local. Municipalities receive taxes for services which affect local welfare such as sanitation, street lighting, local education, water supply etc.

6.4 TAXABLE INCOME AND INCOME TAX

If the income of a person exceeds a certain limit, the excess amount is taxed according to the laid down rules of the government.

The tax so collected is called income tax. The rate of income tax for different groups of earnings is different. In the budget for the year 1988-89 new tariff categories have been introduced:

(i) Income less than Rs. 1 lac;
(ii) Income more than Rs. 1 lac.

I. In the category of income less than 1 lac there are two groups:

(1) Salaried person; and
(2) Non-Salaried person.

The assessment of the two groups will be as below:

(1) *Salaried Person:* Supposing there is an income of Rs. 99,999, the tax will be calculated as follows:

Income	Rs. 99,999
Tax rebate	Rs. 40,000
Taxable income	Rs. 59,999
Tax @ 10%	Rs. 5,999.90

(2) *Non-Salaried Person:* Supposing there is an income of Rs. 99,999, the tax will be calculated as follows:

Income	Rs. 99,999
Tax rebate	Rs. 30,000
Taxable income	Rs. 69,999
Tax @ 10%	Rs. 6,999.90

No investment allowance is admissible under the new rules.

II. For income of Rs. 1 lac or more the rates will be as follows:

Income	*Rebate*	*Taxable Income*	*Tax*
Rs. 1,00,000	Rs. 24,000	Rs. 76,000	Rs. 11,500
		Rs. 1,00,000	Rs.11,500 + 25% of amount exceeding Rs. 76,000
		Rs. 2,00,000	Rs.17,500 + 35% of amount exceeding Rs. 1,00,000
		Exceeding Rs. 2,00,000	Rs.52,500 + 45% of amount exceeding Rs. 2,00,000

(*a*) A tax relief is obtainable for the following types of investments:

(i) Defence Savings Certificate or NIT units.

(ii) Premium of a life insurance policy.

(iii) Charity and donations.

(iv) Investment in Government recognised Companies.

Maximum limit of relief in tax on the above mentioned investments is $33 \frac{1}{3}\%$ of total income or up to Rs. 50,000 (whichever is less). Below is the formula for the tax relief on investments:

$$\text{Tax relief on investments} = \frac{\text{Total income tax} \times \text{Total investments}}{\text{Total income}}$$

Payable income tax = Total income tax - Tax relief in investments.

(*b*) On an income of Rs. 2,00,000 or more an additional surcharge of 10% of total income tax will be charged.

Note: Amount of Zakat has to subtracted from the total income to arrive at the taxable income.

6.5 PROPERTY TAX

The owners of houses, shops and bungalows have also to pay tax. Such a tax is called 'Property Tax'. This tax is levied on an estimated annual income from the property according to the prescribed rates.

6.6 CUSTOM DUTY

Customs duty is charged on goods imported from foreign countries. There are different rates of custom duty on different goods. Government keeps on changing these rates.

6.7 OTHER TAXES

Besides income tax, revenue and custom duty, the Government imposes many 'other taxes as well. For example recreation tax is charged on cinema tickets. Taxes are also charged on use of certain things which are charged in the form of licence fee. For example licence fees are charged on the use of Radio, Television and vehicles, etc. The following examples illustrate the method of finding taxes.

Example 1

Tariq's salary is Rs.6,500 p.m. Calculate the amount of income tax he will have to pay in a year.

Solution:

Monthly income = Rs. 6,500

Total income = Rs. 6,500 × 12

= Rs. 78,000 p.a.

Tax rebate = Rs. 40,000

Taxable income = Rs. 38,000

Income tax payable = 10% of Rs. 38,000

$= \frac{10}{100} \times 38{,}000 = 3{,}800$

= Rs. 3,800

Example 2

Amjad earns Rs. 1,80,000 from a private business in one year. He has invested Rs. 30,000 in defence saving certificates. Calculate the amount of income tax he will have to pay that year.

Solution:

Total income = Rs. 1,80,000

Tax rebate = Rs. 24,000

Taxable income = Rs. 1,56,000

Taxable income in excess of Rs.1 lac = Rs. 56,000

35% of 56,000 = $\frac{35}{100} \times 56,000$

= Rs. 19,600

Income tax = Rs. 17,500 + Rs. 19,600

= Rs. 37,100

Total investments = Rs. 30,000

Tax relief = $\frac{37,100 \times 30,000}{1,80,000}$

= Rs. 6,183.33

Income tax payable = Rs. 37,100 - 6,183.33

= Rs. 30,286.67

Example 3

Owner of construction company makes a profit of Rs. 3,60,000 in one year. He invested Rs. 40,000 in defence savings certificate and donated Rs. 10,000 to a hospital. Calculate the amount of income tax he will have to pay that year.

Solution:

Total income = Rs. 3,60,000

Tax rebate = Rs. 24,000

Taxable income = Rs. 3,36,000

Taxable income in excess of Rs. 2 lac = Rs. 1,36,000

45% of Rs.1,36,000 = $\frac{45}{100} \times 1,36,000$

= Rs. 61,200

Income tax = Rs. 52,500 + Rs. 61,200

= Rs. 1,13,700

Additional surcharge = 10% of Rs. 1,13,700

= Rs. 11,370

Total income tax = Rs. 1,13,700 + 11,370

= Rs. 1,25,070

Total investment = Rs. 40,000 + 10,000

= Rs. 50,000

Tax relief = $\frac{50{,}000 \times 1{,}25{,}070}{3{,}60{,}000}$

= Rs. 17,370.833 = Rs. 17,370.84

Tax payable = Rs. 1,25,070 - 17,370.84

= Rs. 1,07,699.16

Example 3

A motor car reached Karachi in 22500 rupees. What did it cost to the dealer if 180% is paid as custom duty and 2000 rupees are paid as other expenses ?

Solution

Custom duty on 22500 at the rate of 180%

$$= \frac{22500 \times 180}{100} = 40{,}500 \text{ rupees}$$

Other expenditure = 2,000 rupees

Total price of the motor car =

= 22500 + 40500 + 2000 = 65,000 rupees

Example 4

A person gave a house and 3 shops on rent. The rent for the house is 400 rupees per month and the rent for shop is 60 rupees per month. Find property tax at the rate of 10 percent.

Solution

Annual income from house = 12 × 400 = 4,800 rupees

Annual income from 3 shops = 3 × 12 × 60 = 2,160 rupees

Total income = 4800 + 2160 = 6960 rupees

Property tax at the rate of 10%

$$= \frac{6960 \times 10}{100} = 696 \text{ rupees}$$

EXERCISE NO.6.2

1. Annual savings of a person are 2,400 rupees. His wife has ornaments worth Rs. 2,050. What amount of Zakat is due from him?

2. A person paid 56.50 rupees as Zakat in a year. Find his savings for the year.

3. Annual income of a person from salary is Rs. 72,000. His income from other sources is Rs. 6,400. He does not make any investment during the year. Find his income tax for the year.

4. The annual income of a shopkeeper is Rs. 1,04,000. His annual contributions to Provident Fund, Benevolent Fund and Group Insurance amount to Rs. 6,240. Find the amount of income tax to be paid by him.

5. Monthly salary of Ihtesham is Rs.86,000. His annual income from other sources is Rs. 34,000. He pays Rs. 2,000 as Zakat. His annual contributions to different funds amount to Rs. 4,000. He pays Rs. 2,000 as annual insurance premium. Find his income tax for the year.

6. The rent of a house is 150 rupees per month. Find property tax due on him at the rate of 15 percent.

7. The rent of a house is 5 percent of its value. Property tax is 15 percent of its rental value. Find the price of the house if its owner pays Rs.165 as property tax.

8. Thr price of a radio was 275 rupees at the Karachi Sea Port. Find the inland price of the radio if the custom duty was paid at the rate of 80 percent.

9. A person sent a motor car from outside Pakistan. He paid 21600 rupees for it upto Karachi. On the amount he had to pay 80 percent as custom duty and a sum of 425 rupees as insurance. Find the total inland price of the motor car.

10. A person brought some goods worth Rs.4,500 from outside Pakistan. He paid 25 percent as custom duty. He sold the goods to a shopkeeper at 20 percent profit. Find the shopkeeper's buying price.

11. Ayesha had ornaments worth Rs.25000 and some cash for the whole year. She paid Rs.740 as Zakat at the end of the year. How much cash did she have ?

12. On cinema tickets 25 percent is charged as recreation tax. In a cinema tickets worth Rs.4,250 were sold. What did the owner get and how much was paid as recreation tax ?

6.8 DISCOUNT, REDUCTION

When the goods are on clearance sale, the shopkeepers sell them at reduced rates. Some shopkeepers advertise percentage reduction in the prices. Sometimes the customers ask the shopkeepers for reduction in the demanded prices. This reduction is called 'Discount'. See examples below for explanation.

Example 1

A shopkeeper announced 15 percent reduction in the price of wool on clearance sale. Find the reduced price of wool if its previous price was 160 rupees per kilogram.

Solution

Price of wool before clearance sale = 160 rupees per kilogram.

Discount = 15%

Total reduction in the price $= \frac{160 \times 15}{100} = 24$ rupees per kg

∴ Reduced price = 160 – 24 = 136 rupees per kg.

Example 2

On a clearance sale, the price of a bag was reduced from 64 rupees to 48 rupees. Find the discount percent.

Solution

Written price = 64 rupees

Reduced price = 48 rupees

Total reduction = 64 – 48 = 16 rupees

Discount % $= \frac{16}{64} \times 100 = 25$

∴ Discount = 25%

6.9 COMMISSION, BROKERAGE

Some trading companies write the retail price on their manufactured goods. So a shopkeeper cannot charge a price more than the written price. You know that nobody can run a business without getting profit out of it. Therefore, the trading companies allow some reduction in the written price to the shopkeepers who sell their manufactured goods. This reduction is called 'Commission'. Property dealers are commission agents and usually charge comission both from the seller and the buyer. In Pakistani markets it is customary to charge brokerage in buying and selling of trade goods. Examples are given below for explanation :

Example 1

A factory owner gives 15% commission on the marked price to a shopkeeper. In case of cash payment, he gets an additional commission of 5% on the balance after deduction of 15% commission. What does the shopkeeper pay in cash for an article marked price of which is Rs.1200 ?

Solution

Marked price = 1,200 rupees

Commission at the rate of 15% $= \dfrac{15 \times 1200}{100} = 180$ rupees

Remaining price after the deduction of 15% commission =

$$1200 - 180 = \text{Rs.}1020$$

Commission at the rate of 5% on Rs.1020 for cash payment.

$$= \frac{5 \times 1020}{100} = \text{Rs.}51$$

Payment to be made by the shopkeeper = 1020 − 51 = Rs.969

Example 2

A commission agent got a house sold for 35,000 rupees and charged 2 percent commission from both the parties. What did the seller get ? What did the buyer pay ?
What did the agent get ?

Solution

The price of the house = 35,000 rupees

Agent's commission from each party, the seller and the buyer $= \dfrac{2 \times 35000}{100} = 700$ rupees

The seller got 35000 − 700 = 34,300 rupees

The buyer paid 35000 + 700 = 35,700 rupees

Agent's commission = 700 + 700 = 1,400 rupees

Example 3

Jaffer is a salesman in a shop. He gets 200 rupees as monthly pay and 5 percent commission on the price of the goods sold. He sold goods worth Rs.4,500 in a month. Find his income for the month.

Solution

The price of the goods sold = 4,500 rupees

Commission at the rate of 5% = $\frac{5 \times 4500}{100}$ = Rs.225

Jafar's monthly pay = 200 rupees

jafar's total income for the month = 200 + 225 = Rs.425

Example 4

A broker sold rice of a landlord worth Rs.12,000 . He deducted 7½% as brokerage. What did the broker get ? If he had paid 12,950 rupees to the landlord after deducting his brokerage at the same rate, what would have been the actual selling price of the rice ?

Solution

Selling price of rice = 12,000 rupees

Brokerage at the rate of 7½ % = $\frac{15}{2} \times \frac{12000}{100}$

= 900 rupees

∴ The broker got 900 rupees.

Brokerage = 7½%

For every 100 rupees the landlord gets

$$100 - \frac{15}{2} = \frac{200-15}{2} = \frac{185}{2} \text{ rupees}$$

The landlord gets $\frac{185}{2}$ rupees if the actual selling price of rice = Rs.100

He gets 12,950 rupees if the actual selling price of rice

$$= \frac{100 \times 2 \times 12950}{185} = 14{,}000 \text{ rupees}$$

EXERCISE NO.6.3

1. The marked price of an article is 450 rupees. A discount of 20 percent is given on the marked price. Find the discount and the reduced price of the article.

2. Samad placed an order for 50 books. The written price of every book is Rs.3.20 and 20 percent discount is allowed. Find the discount and the reduced price of the books.

3. Jamal Khan purchased the following articles and was allowed discount at different rates. How much did he pay in total?

(a) Books of 550 rupees at 20 percent discount.

(b) Note Books of Rs.250 at 15 percent discount.

(c) Pens of 120 rupees at 10 percent discount.

4. Aslam bought cloth at 15 percent discount. He also got 5 percent reduction by paying in cash. What did he pay for the cloth, the marked price of which is 80 rupees?

5. In a branch of a shoe company an agent gets 240 rupees per month and 10 percent commission on total sale. In a month the sale in his shop amounted to 7,545 rupees. Find his total income for the month.

6. A newspaper seller sold newspapers of 540 rupees in a month and got a total commission of 81 rupees. Find the percentage of his commission.

7. A shopkeeper writes 25 percent more than the original price of things. Then he announces 16 percent discount on sale. Find the price of an article on sale, the original price of which is 180 rupees.

8. A factory owner allows 15 percent commission on goods of the value upto Rs.15000 and gives 20 percent commission on the excess value beyond Rs.15000. An agent bought goods worth Rs.26,500. Find his commission.

9. A broker sold Sarwar's plot to Jamil for 11,000 rupees. He got 1 percent commission from both the parties. Find broker's total commission. What did Sarwar get ? What did Jamil pay ?

10. A broker sold rice. He charged 12½ percent brokerage from the landlord. What brokerage did he get if he sold rice worth Rs.10,560. ?

6.10 BUYING AND SELLING ON INSTALMENTS

Nowadays in many countries the buying and selling of things on instalments is becoming common. This is happening under necessity. In the present times the standard of living of the people is rising. Therefore, many people want to have fine furniture, sewing machine, cycle, radio and T.V. etc. Since some people have limited sources of income, therefore, they are unable to pay the prices of things in full and so they buy these things on instalments. In such a business since the shopkeeper does not get the whole amount in cash, therefore, he sells the things on prices higher than the usual. See the example for explanation.

Example

A ceiling fan costs Rs.600. Jamil had only 120 rupees for cash payment. The shopkeeper asked him to pay the remaining price inclusive of simple Interest in 6 monthly instalments of Rs.87 each. Find the excess amount paid by him. Find also the rate of interest .

Solution

Price of the fan = Rs.600
Cash payment = Rs.120
Balance = Rs.480

If only 480 rupees were to be paid in six monthly instalments, Jamil would have paid 80 rupees per month. But he pays a monthly instalment of Rs.87.

∴ Amount paid in 6 instalments = 87 x 6 = Rs.552

Amount paid in excess of the actual price
= Rs.522 -- Rs.480 = Rs.42

Debt in the first month = Rs.480

Debt in the last month = Rs.80

$$\text{Average debt per month} = \frac{480 + 80}{2} = \text{Rs.}280$$

Interest paid on Rs.280 in 6 months = Rs.42

$$\text{Rate of interest percent per annum} = \frac{\text{Interest} \times 100}{\text{Principal} \times \text{time}}$$

$$= \frac{42 \times 100}{280 \times \frac{6}{12}} = \frac{42 \times 100 \times 12}{280 \times 6} = 30$$

6.11 SOCIETIES' AND BANKS' LOANS

If you need money for agricultural puproses, for purchasing land or for house building , the banks, societies and other financial organisations provide loans. Interest at fixed rates is paid on such loans. See the following example for illustration :

Example

Majid got a bank loan of 10,000 rupees. The rate of interest was fixed to be 10 percent which was to be calculated six-monthly Majid pays six-monthly instalments of 2,500 rupees. What amount of interest will be pay ? The amount of interest is to be paid with the last instalment.

Solution

The amount of loan = 10,000 rupees

The amount of loan after 6 months

$$= 10000 + \frac{10000 \times 10}{100} \times \frac{1}{2}$$

$$= 10000 + 500 = 10{,}500 \text{ rupees}$$

First instalment = 2,500 rupees

The amount of loan after paying the first instalment

$= 10500 - 2500 = 8,000$ rupees

The amount of loan after one year

$$= 8000 + \frac{8000 \times 10}{100} \times \frac{1}{2}$$

$= 8000 + 400 = 8400$ rupees

The amount of loan after paying the 2nd instalment

$= 8400 - 2500 = 5900$ rupees

The amount of loan after one year 6 months

$$= 5900 + \frac{5900 \times 10}{100} \times \frac{1}{2} = 5900 + 295$$

$= 6,195$ rupees

The amount of loan after paying the third instalment

$= 6195 - 2500 = 3,695$ rupees

The amount of loan after two years

$$= 3695 + \frac{3695 \times 10}{100} \times \frac{1}{2}$$

$$= 3695 + \frac{739}{4} = 3695 + 184.75 \text{ rupees}$$

$= 3879.75$ rupees

The amount of loan after paying the fourth instalment

$= 3879.75 - 2500 = 1379.75$ rupees

The amount of loan = 10,000 rupees

The amount paid in 4 instalments

$= 2500 \times 4 = 10,000$ rupees

The amount payable after paying the 4th instalment

$= 1,379.75$ rupees

Hence the amount of interest $= 1,379.75$ rupees

EXERCISE NO.6.4

1. The cash price of a T.V. set is Rs.4000 . The dealer sells it to a customer on cash payment of Rs.400 and changes 12% additional profit on the remaining price. In lieu thereof, he provides him the facility to pay the remaining amount in 9 monthly instalments. Find the amount of monthly instalment.

2. The cash price of a machine is 650 rupees. The machine can be purhased by paying 50 rupees in cash. The shopkeeper takes 24% additional profit on the remaining price and agrees to the payment of the amount in 24 monthly instalments. Find the amount of monthly instalment.

3. Shamim bought a fan, the cash price of which is 660 rupees. He paid 60 rupees in cash and the remaining amount was paid in 10 monthly instalments of Rs.64.40 each. How much did he pay more ? What was the rate of interest per year ?

4. The cash price of a machine is 700 rupees. Aslam bought that machine by paying 100 rupees in cash and 10 monthly instalments. If the monthly instalment is Rs.68.80, find rate of interest percent.

5. The cash price of a cycle is 800 rupees. Find the percent rate of interest if it was bought by paying 100 rupees in cash and 14 instalments of Rs.60 each. What amount was paid in excess ?

6. Ashfaq took a loan of Rs.20,000 . The rate of interest is 10 percent per year. The loan is to be paid back in 4 half-yearly instalments. Each instalment is to include ¼ of the loan along with the interest on the loan due upto the time of payment of the instalment. Find the instalments.

7. A person took a loan of Rs.12,000 from a society. He promised to repay the loan in 6 six-monthly instalments. Find the amount of each instalment when the mode of payment is such that an instalment includes $\frac{1}{6}$ of the loan and interest due on the loan upto the payment of the instalment. The rate of interest is 10 percent per annum.

8. A person took a loan of Rs.5000 from Zamindara Bank. He promised to pay back the loan in 2 six-monthly instalments. Find the amount of each instalment when an instalment is to comprise half the loan and the interest due on the loan at the time of payment of the instalment ; the rate of interest being 10 percent per annum.

CHAPTER 7

FUNDAMENTAL OPERATIONS ON ALGEBRAIC EXPRESSIONS

Before explaining the laws of exponents, it is necessary to explain a few terms.

7.1 VARIABLE

A variable is a symbol which denotes every member of a specified set. For example in the set $A = \{ x \mid 1 \leqslant x \leqslant 10 \}$, x is a variable which denotes every number from 1 to 10.

The letters x, y, z are used for denoting variables.

N. B. Every member of a specified set is called a CONSTANT.

7.2 ALGEBRAIC EXPRESSIONS

Consider the following expressions :

(i) x (ii) -5

(iii) $3x + 2y - \dfrac{5}{z}$ (iv) $\sqrt{x} - \dfrac{1}{\sqrt{x}}$

(v) $x^2 + x - 1$

Each one of these expressions is either a variable or a constant i.e. real number, or, has been obtained from constants and variables with the help of operations of addition, subtraction, multiplication, division, root extraction, and raising to powers. Such expressions are called ALGEBRAIC EXPRESSIONS.

7.3 TERMS OF AN EXPRESSION

The parts of an algebraic expression separated from one another by + or − signs, are called its terms. For example, the algebraic expression '$6y$' contains only one term, the expression $2x - 7$ has two terms, and the expression $x^2 + y^2 - \sqrt{z}$ has three terms.

7.4 CO-EFFICIENTS

The constant factor of a term of an expression is called the CO-EFFICIENT of that term. For example, in the expression $3x - 9y$, the co-efficient of x is 3 and the co-efficient of y is −9.

7.5 POLYNOMIAL

In the beginning of the Chapter you have read about variables constants and algebraic expressions. If an algebraic expression is a constant or a variable or consists of terms obtained from constants and variables by the operations of addition, subtraction or multiplication, it is called a POLYNOMIAL. It is clear that in a polynomial, the exponent of any power of a variable must be a positive integer.

If a polynomial consists of a single term, it is called a MONOMIAL, e.g., each of the expressions 6, $3x$, 0 is a monomial.

A polynomial consisting of two terms is called a BINOMIAL, e.g., each of the expressions $2x - 3y$, $\sqrt{3x^2} - Y^2$ is a binomial.

N.B. According to the definition of a polynomial the following expression are not polynomials :

(i) $\dfrac{1}{y}$ (ii) $x + \dfrac{1}{x}$ (iii) $-\dfrac{5}{x^2}$

(iv) $\sqrt{x}$

7.6 ASCENDING AND DESCENDING ORDERS OF A POLYNOMIAL

It is often useful to write a given polynomial in a certain order, descending or ascending order.

Consider the following polynomials :

(i) $y^3 + 5y^2 + y + 2$ (ii) $2 + y + 5y^2 + y^3$

(iii) $y^3 + 2 + 5y^2 + y$

The first polynomial contains four terms, first term is y^3, second $5y^2$, third y and fourth 2.

We notice that the exponents of the variable y go on decreasing, i.e. they are respectively 3, 2, 1, 0

Such an arrangement of an expression is called DESCEDING ORDER (descend' means to come down', 'to decrease')

In the second expression the first term is 2 $(2y^0)$, second y, third $5y^2$ and fourth y^3. Here the exponent of the variable y assume successively the value 0, 1, 2, 3. Such an arrangement of an expression is called ASCENDING ORDER ('ascend' means 'to go up', ' to increase'.

In the **third** expression there are four terms. First being y^3, second 2 or $2y^0$,* third $5y^2$ and fourth y. We see that in this expression the exponents of the terms (3, 0, 2, 1) are in a deranged or unordered form.

An unordered polynomial may be arranged in an ascending or descending order by suitably rearranging its terms.

7.7. ADDITION OF POLYNOMIALS

In the lower classes you have studied commutative and associative properties of addition and distributive properties of multiplication. We solve the following examples by using them.

Example 1

Simplify $7x^3 + 5x^2 + 6x^3$

Solution

$$7x^3 + 5x^2 + 6x^3 = 7x^3 + (5x^2 + 6x^3)$$

(Associative property of addition)

$$= 7x^3 + (6x^3 + 5x^2)$$

(Commutative property of addition)

$$= (7x^3 + 6x^3) + 5x^2$$

(Associative property of addition)

$$= (7 + 6)x^3 + 5x^2$$

(Distributive property of multiplication)

$$= 13x^3 + 5x^2$$

Example 2

Simplify $13y^2 + 5y + 7y^2 + 4y$

Solution

$$13y^2 + 5y + 7y^2 + 4y = (13y^2 + 7y^2) + (5y + 4y)$$

$$= 20y^2 + 9y$$

Example 3

Add $6z^2 - 4 - 3z$, $6z + 7z^2 - 3$, $-9z^2 - 5 - 5z$

Solution

$$\begin{array}{r}
6z^2 - 3z - 4 \\
7z^2 + 6z - 3 \\
\underline{-9z^2 - 5z - 5} \\
\underline{4z^2 - 2z - 12}
\end{array}$$

Explanation

First we arranged the polynomials in descending order, wrote them in such a way that the terms with the same exponents are in the same column and then added according to the following method :

(i) We noted that the sum of positive co-efficients in the first column is 13 and the negative co-efficients is -9. Therefore, the co-efficient of the z^2 term in the answer is 4.

(ii) In the second column the positive co-efficient is 6 and the sum of negative co-efficients is -8. Therefore, the second term of the asnwer is $-2z$.

(iii) All the constant terms in the third column are negative Their sum is -12 which is, therefore, the third term in the answer.

Thus the complete answer is $4z^2 - 2z - 12$.

7.8 SUBTRACTION OF POLYNOMIALS

Subtraction is the inverse operation of addition. Therefore, subtraction of a number 'a' from a given number is equivalent to adding the additive inverse $-a$ to that number. Thus $8 - 5$ is equivalent to $8 + (-5)$.

Example 1

Subtract $10x^3 + 6x^2 - 7x + 7$ from $8x^3 - 9x^2 - x + 7$

Solution

$$\begin{array}{rrrr}
8x^3 & -\,9x^2 & -\,x & +\,7 \\
10x^3 & +\,6x^2 & -\,7x & +\,7 \\
- & - & + & - \\
\hline
-\,2x^3 & -\,15x^2 & +\,6x & \\
\hline
\end{array}$$

Explanation

First we wrote the given expressions one below the other in such a way that terms with the same exponents are in the same column. Then we reversed the signs of the lower terms and added them to the upper terms.

Example 2

Subtract: $3a^5 - 7a^3 + 8a$ from $5a^4 - 7a^5 - 3a^3$

Solution

$$\begin{array}{rrrr}
-7a^5 & +5a^4 & -3a^3 & \\
3a^5 & & -7a^3 & +8a \\
- & & + & - \\
\hline
-10a^5 & +5a^4 & +4a^3 & -8a \\
\hline
\end{array}$$

This example can also be solved thus:

$$\begin{aligned}
& (5a^4 - 7a^5 - 3a^3) - (3a^5 - 7a^3 + 8a) \\
= {} & 5a^4 - 7a^5 - 3a^3 - 3a^5 + 7a^3 - 8a \\
= {} & 5a^4 - 7a^5 - 3a^5 - 3a^3 + 7a^3 - 8a \\
= {} & 5a^4 + (-7a^5 - 3a^5) + (-3a^3 + 7a^3) - 8a \\
= {} & 5a^4 - 10a^5 + 4a^3 - 8a \\
= {} & -10a^5 + 5a^4 + 4a^3 - 8a
\end{aligned}$$

Thus we get the following rule for opening a bracket :

If negative sign precedes a bracket, the signs of all the terms within the brackets will be reversed on removing the brackets.

EXERCISE NO.7.1

1. Write the following expressions in the (a) descending order and (b) ascending order :

(i) $y^3 + y - 2y^2 - 1$;

(ii) $6 - 7x^5 + 8x^4 + x^3$;

(iii) $4a^2 - 3 - 5a - a^4$;

(iv) $10x^2 - 5x^6 + 3 - 2x^4 + 3x$.

2. Add up the expressions :

(i) $5x^3 + 2x^2 - 1$, $3x^3 - 7x^2 + 8$, $x^3 + 2x + 1$;

(ii) $2a - 5 + 6a^2$, $5a^2 - 3a + 8$, $-5a + 3a^2 - 1$;

(iii) $xy + x^2y^2 - x^3y^3$, $-xy - x^2y^2 + x^3y^3$, $xy + x^2y^2$.

3. Subtract

$7a^4 + 10a^5 - 4a^3 + 9$ from $6a^5 - 3a^4 - 7a^3 + 8a$

4. Subtract

$3x^4 - 2x^3 - 1 + 6x^2$ from $4x^4 + 6x^2 + 8 - 3x^3 - x$

5. Simplify

(1) $(2x + 3y - 4z) + (3y - 6x + 5z) + (x \quad y - z)$

(2) $(6a - 3b + 4c - 8d) - (3a + 11b + 3c + 8d)$

6. If $A = 2x^3 - 4x^2 - 5$, $B = 4x^3 - 9x + 8x^2 - 3$,

$C = -6x^2 - 3x^3 + 2 - 5x$ find the value of A + B + C.

7. Subtract $4x^2y^2 - 7x^3y + 8$ from the sum of $3x^3y + 6x^2y^2 - 7xy^3 + 8$ and $-8x^2y^2 + 6x^3y + 8xy^3$

8. Evaluate X + Y − Z when

$X = 8l^5 - 3l^3 + 6l^2 - 1$

$$Y = 7t^4 + 8t^3 - 6t^5 + 2$$

$$Z = -3t^5 - 2t^3 \div 4t^4 + 8 - 2t^2$$

9. Subtract $6a - 9b + 3c$ from the sum of $a + b - c$ and $-a + b - c$.

10. How much greater is the sum of $2x^3 + 7x^2 - 8x$ and $-2x^3 - 7x^2 + 11$ than the sum of $-3 + 7x^2 - 9x$ and $4x^2 - 3x^3 + 5$?

7.9 MULTIPLICATIKN OF POLYNOMIALS

You have studied the Law of Powers in the beginning of this Chapter. For multiplying polynomials, we shall use the Law of Product of Powers. We know that we have to add the exponents for multiplying powers with the same base. For finding the complete product of two monomials we shall have to multiply the numerical co-efficients also.

For example :

(i) $6x^3 \times 5x^4 = 30x^7$ $[\because (+a)(+b) = +ab]$

(ii) $-4pq \times (-3pq^2\} = 12p^2q^3$ $[\because (-a)(-b) = +ab]$

(iii) $-2y^2z^2 \times 7y^3z^4 = -14y^5z^6$ $[\because (-a)(+b) = -ab]$

Example

Multiply $2x^2 - 3x + 4$ and $3x^2 - 4x - 2$.

Solution

$$\begin{array}{lllllll}
2x^2 & -3x & +4 & & & & \\
\underline{3x^2} & \underline{-4x} & \underline{-2} & & & & \\
6x^4 & -9x^3 & +12x^2 & & & & \text{(Product of first expression and } 3x^2\text{)} \\
 & -8x^3 & +12x^2 & -16x & & & \text{(Product of first expression and} -4x\text{)} \\
 & & \underline{-4x^2} & \underline{+6x} & \underline{-8} & & \text{(Product of first expression and} -2\text{)} \\
\underline{6x^4} & \underline{-17x^3} & \underline{+20x^2} & \underline{-10x} & \underline{-8} & &
\end{array}$$

EXERCISE NO.7.2

Find the product :

1. $(x+3)(x+3)$ 2. $(y+9)(y-9)$
3. $(a+b)(a^2-ab+b^2)$ 4. $(2x-3y)(4x^2+9y^2+6xy)$
5. $(x+y-z)(x-y+z)$ 6. $(l^2+l+1)(l^2-l+1)$
7. $(5a-3+2a^2)(1-3a^2+2a^4-a)$
8. $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$
9. If $A=2p-3q-4r$ and $B=4q+5p-2r$, find the value of $3A-4B$
10. Evaluate $-3P+2Q-R$, when

$$P=-3x^3+4x^2-1,$$
$$Q=-7x+2x^3-8$$
$$R=x^3-x^2+x-1$$

7.10 DIVISION OF POLYNOMIALS

In the process of divison we use the Law of Quotient of Powers. The law is also called the Law of Powers for Division. We know that according to this law we have to subtract the exponents for dividing powers with the same base.

For performing the operation of division it is essential to remember the following laws of sings :

(1) $\dfrac{+a}{+b}=+\dfrac{a}{b}$, (2) $\dfrac{-a}{-b}=+\dfrac{a}{b}$,

(3) $\dfrac{+a}{-b}=-\dfrac{a}{b}$, (4) $\dfrac{-a}{+b}=-\dfrac{a}{b}$

Example 1

Divide $16x^5-24x^4+12x^3-4x^2$ by $4x^2$ when $(x\neq 0)$

Solution $\dfrac{16x^5-24x^4+12x^3-4x^2}{4x^2}=\dfrac{16x^5}{4x^2}-\dfrac{24x^4}{4x^2}+\dfrac{12x^3}{4x^2}-\dfrac{4x^2}{4x^2}$

$= 4x^{5-2} - 6x^{4-2} + 3x^{3-2} - x^{2-2}$ (by the law of quotient of Powers)

$= 4x^3 - 6x^2 + 3x - 1 \qquad (x^0 = 1)$

The example shows that if the divisor contains only one term and the dividend contains one or more terms, the operation of division can be performed by dividing each of the terms of the dividend and then combining the results.

Example 2

Divide

$$6x^4 + 5y^4 - 13x^3y - 13xy^3 + 23x^2y^2 \text{ by } 3x^2 + y^2 - 2xy$$

Solution

$$\begin{array}{r|llllll}
 & 2x^2 & -3xy & +5y^2 & & \\ \cline{2-6}
3x^2-2xy+y^2 & 6x^4 & -13x^3y & +23x^2y^2 & -13xy^3 & +5y^4\\
 & \underset{-}{6x^4} & \underset{+}{-4x^3y} & \underset{-}{+2x^2y^2} & & \\ \cline{2-4}
 & & -9x^3y & +21x^2y^2 & -13xy^3 & \\
 & & \underset{+}{-9x^3y} & \underset{-}{+6x^2y^2} & \underset{+}{-3xy^3} & \\ \cline{3-5}
 & & & 15x^2y^2 & -10xy^3 & +5y^4\\
 & & & \underset{-}{15x^2y^2} & \underset{+}{-10xy^3} & \underset{-}{+5y^4}\\ \cline{4-6}
 & & & & 0 & \\ \cline{4-6}
\end{array}$$

Explanations

(1) First we wrote the dividend and the divisor in descending powers of x.

(2) By dividing the first term of the dividend, $6x^4$, by the first term of the divisor $3x^2$, we obtain $2x^2$, the first term of the quotient.

(3) By multiplying all the terms of the divisor $3x^2 - 2xy + y^2$ by $2x^2$, we get $6x^4 - 4x^3y + 2x^2y^2$ which has been subtracted from the dividend.

(4) With the remainder we combine the next term of the dividend, and perform the operation of division by the divisor. The operation of division is continued in the same manner, till we get zero as the remainder or it is an expression of a lesser degree than the divisor.

N.B. By the DEGREE of a polynomial in a single variable we mean the exponent of the variable in the term involving the largest power of that variable. For example the degree of $4x + 3x^2 + 1$ is 2, and of $ax + b$ it is 1.

If a polynomial involves two or more variables, its degree is determined by considering the sum of the exponents in each term separately. Thus the polynomial $ax^2 + bxy + cxz^2$ is of the third degree.

EXERCISE NO.7.3

Simplify :

1. $(25x^3y^4) \div (5xy^3)$ 2. $(-121a^6b^3) \div (-11a^3b^3)$

3. $(16y^8z^5 - 48y^7z^6 - 144y^3y^4) \div (8y^2z^4)$

4. $(x^2 - 5x + 6) \div (x - 2)$ 5. $(x^4 - y^4) \div (x - y)$

6. $(6x^2 - 2x - x^4 - 4x^3 + x^5) \div (x^3 - 4x + 2)$

7. $(x^3 - 64) \div (x^2 + 4x + 16)$ 8. $(-16x^4 + 1) \div (4x^2 - 1)$

9. $(2 - 4y^2 + 12y^4 + 7y - 17y^3) \div (2y + 1 - 3y^2)$

10. $(a^3 + b^3 + c^3 - 3abc) \div (a + b + c)$

11. Divide $x^6 + 4x^5 - 16x^3 + x + 2x^2 + 3 - 3x^4$

by $x^3 + 4x^2 + 1 + 2x$

13. For what value of q is the expression

$5x^3 - 14x + q$ exactly divisible by $x - 2$.

CHAPTER 8

FORMULAE AND FACTORS

8.1 Formulae

In the preceding Chapter we have learnt the method of multiplying algebraic expressions. Some results of multiplication are repeatedly used in Algebra. Such results are called FORMULAE. By using them we can find out the product of certain expressions without performing the operation of multiplication. In the solution of sums, their use is of much convenience, and it saves a lot of time and labour. A formula is an equation which is true for all the values of the variables involved in it. On this account a formula is also called an IDENTITY.

For example in the formula

$$(a + b)^2 = a^2 + 2ab + b^2 \quad \forall\ a, b$$

If we substitute any numerical values for a and b the two sides of the equation will always be equal.

You have already studied a few formulae, including the one above, in Class VIII. Here we shall revise those formulae, use them in solving questions, and study some new formulae as well.

8.2 FORMULA I

$$k(a + b + c) = ka + kb + kc \quad \forall\ k, a, b, c$$

Algebraic Proof

Let $\quad b + c = d$

$$k(a + b + c) = k(a + d) \quad \forall\ k, a, b, c$$

$= ka + kd$ (Distributive property of multiplication w.r.t. addition).

$= ka + k(b + c)$, Replacing the value of d)

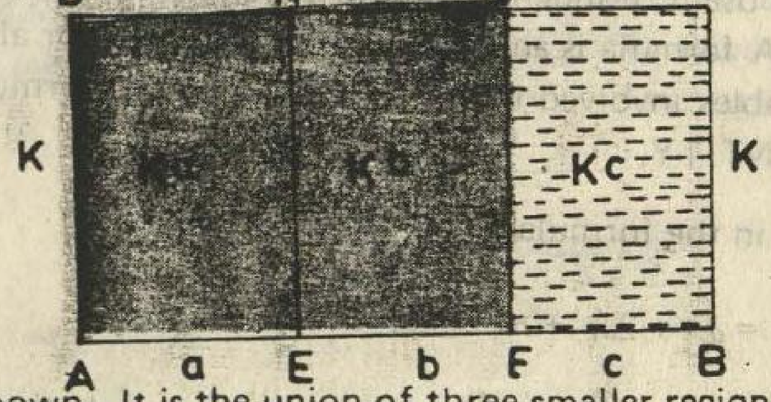
$= ka + kb + kc$ (Distributive property of multipli
tion w.r.t. addition)

Thus

$$k(a + b + c) = ka + kb + kc \quad \forall k, a, b, c$$

Geometrical proof

In the adjoining figure a rectangular region ABCD has been



shown. It is the union of three smaller regions.

$m\overline{AB} = a + b + c$, $m\overline{AD} = k$, $m\overline{AE} = a$, $m\overline{EF} = b$, $m\overline{FB} = c$

Area of rectangular region ABCD = area of rectangular regio
AEHD + area of rectangular region EFGH + area of rectangular
region FBCG.

$$\therefore \quad k(a + b + c) = ka + kb + kc$$

Example 1

Find the product

$$2ab^2(6a^2b + 3ab + 4b^2c)$$

Solution

Given expression

$$= 2ab^2(6a^2b + 3ab + 4b^2c)$$

$$= (2ab^2)(6a^2b) + (2ab^2)(3ab) + (2ab^2)(4b^2c)$$

$$= 12a^3b^3 + 6a^2b^3 + 8ab^4c.$$

Example 2

Find the product

$$-3x^2yz(2x^3y - 3x^2z + 4y^3z^3)$$

Solution

Given expression

$$= -3x^2yz(2x^3y - 3x^2z + 4y^3z^3)$$

$$= -6x^5y^2z + 9x^4yz^2 - 12x^2y^4z^4$$

N. B. Remember the following rules of signs which are often used in multiplication :

(i) $(+a)(+b) = +ab$ (ii) $(+a)(-b) = -ab$

(iii) $(-a)(-b) = +ab$ (iv) $(-a)(+b) = -ab$

i.e. to say the product of like signs is positive

e.g., $(2ab^2)(6a^2b) = 12a^3b^3$, $(-3x^2yz)(-3x^2yz) = 9x^4y^2z^2$

and the product of terms with unlike signs is negative,

e.g. $(-3x^2yz)(2x^3y) = -6x^5y^2z$

$(4y^3z^3)(-3x^2yz) = -12x^2y^4z^4$

EXERCISE NO. 8.1

Simplify the following expressions :

(The first five questions should be done orally).

(1) $m(a + b)$ 2. $x(a + x)$

3. $b(b - d)$ 4. $7(3a - 4b)$

5. $4a(2a - 3b + 4c)$ 6. $3x(xy - yz - zx)$

7. $5a(2a^2 - 5a + 1)$ 8. $x^2(3y - 2z - 4x^3)$

9. $-4(3p - 2q + 4r)$ 10. $-5ab(a^2b + 2ab^2 - 3c)$

11. $a(a + b) + c(a + b)$ 12; $d(m + n) + n(c - d)$

13. $ab(bc - cd) - ac(b^2 + bd)$

14. $x(x + y) - 1(x + y)$ 15. $lm(l + m) - l(l - m)$

16: $abc(ab^2c + abc^2 + a^2bc)$

17. $3xyz(x^2yz + xy^2z - xyz^2)$

18. $-4a^2k(5akc - 3ak^2 + 2kb)$

19. $x^n(a^n + b^n - c^n)$ 20. $a^m(a^m - b^n + c^m)$

8.3 FACTORS AND FACTORISATION

We know that if two or more quantities are multiplied, the result is called their PRODUCT and the quantities so multiplied are called the FACTORS of the product. Accordingly, the product of 2 and 15 is 30 and 2 and 15 are factors of 30. Similarly, 1, 30 and 2, 3, 5 are also factors of 30. As each of the members of the set { 1, 2, 3, 5, 6, 10, 15, 30 } divides 30 exactly, this is the set of DIVISORS of 30. The process of expressing a number as a product of factors is called FACTORISATION. A few factorisations of 30 are given below :

(i) 30 = 1 × 30

(ii) 30 = 2 × 15

(iii) 30 = 3 × 10

(iv) 30 = 2 × 3 × 5

In the last factorisation each factor is a prime number. This is the PRIME FACTORISATION of 30. Each of the other factorisations contains a composite factor. It should be remembered that

Every number has one and only one prime factorisation.

We can factorise algebraic expressions also. e.g. $6a^2b^3m = 2 \times 3 \times a \times a \times b \times b \times b \times m$. But it is not quite easy to factorise compound expressions.

Just as multiplication tables are used in factorising composite numbers, we use algebraic rules , i.e., formulae for factorising algebraic expressions. Accordingly, in this Chapter we shall consider factorisation of algebraic expressions with the help of formulae as well.

8.4 FACTORISATION OF EXPRESSIONS OF THE FORM ka + kb + kc.

We know that

$$k(a + b + c) = ka + kb + kc \ \forall a, b, c, k$$

Therefore, interchanging the two sides of this identity,

$$ka + kb + kc = k(a + b + c), \quad \forall k, a, b, c$$

i.e., k and $a + b + c$ are the factors of $ka + kb + kc$

For factorising compound expressions we use the distributive property of multiplication. If a factor occurs in all the terms of the expression, it is a factor of the whole expression. Thus k is a factor of the expression $ka + kb + kc$. The other factor of the expression is clearly obtained by dividing all the terms by k, i.e., the other factor is

$$\frac{ka}{k} + \frac{kb}{k} = \frac{kc}{k} = a + b + c$$

Example 1

Factorise $6x^3 - 12x^2 + 18x$

Solution

Here $6x$ is a factor which is common to all the terms.

$$\therefore 6x^3 - 12x^2 + 18x = 6x\left(\frac{6x^3}{6x} - \frac{12x^2}{6x} + \frac{18x}{6x} \right)*$$

$= 6x(x^2 - 2x + 3)$

**N.B.* (1) After some practice the intermediate step may be omitted.

(2) The common factor is written as a whole and it is not further factorised. Thus in the above example $6x$ is not written as $2 \times 3 \times x$.

Example 2

Factorise: $2a^3b + 6a^2b^2 - 4ab^3$

Solution

Given expression

$$= 2a^3b + 6a^2b^2 - 4ab^3$$

$$= 2ab(a^2 - 3ab + 2b^2)$$

Example 3

Factorise $a(x+y) - b(x+y)$

Solution

Let $x + y = m$. Then

Given expression

$$a(x+y) - b(x+y) = am - mb = m(a-b)$$

$$= (x+y)(a-b) \text{ (Replacing the value of } m)$$

Example 4

Factorise $ax + bx + ay + by$

[ANALYSIS : No factor is common in all the terms of the given expression. To factorise such expressions we partition it into groups such that : (i) each group has a common factor, and (ii) the other factor of each group is the same. Solution of the example in hand will clarify the procedure].

Solution

Expression $= ax + bx + ay + by$

$= (ax + bx) + (ay + by)$

$= x(a + b) + y(a + b)$

$= (a + b)(x + y)$

Example 5

Factorise $4x^2 + 3y + 3xy + 4x$

Solution

The first two terms of this expression do not contain any common factor . Therefore, we combine the first term with the last and the second with the third. Thus

Given expression

$= 4x^2 + 3y + 3xy + 4x$

$= (4x^2 + 4x) + (3y + 3xy) = 4x(x + 1) + 3y(1 + x)$

$= 4x(x + 1) + 3y(x + 1)$

$= (x + 1)(4x + 3y)$

We solve this example again by combining the first term with the third and grouping the remaining two terms.

Given expression

$= (4x^2 + 3xy) + (3y + 4x) = x(4x + 3y) + 1(3y + 4x)$

$= x(4x + 3y) + 1(4x + 3y)$

$= (4x + 3y)(x + 1)$

EXERCISE NO. 8.2

Fractorise (the first ten questions should be answered orally) :

1. $ab + ac$ 2. $bx + xy$ 3. $a^2 - 2a$

4. $xy + y^2$ 5. $ax + x^2$ 6. $15x^2 + 10xy$

7. $-4c - 16$ 8. $-12x^2 + 24x$ 9. $2x^2 - 2x$

10. $xy + yz - y^2$ 11. $4ab - 8ac - 6ax$

12. $6x^2yz + 9xy^2z + 12xyz^2$

13. $5x^3y + 15x^2y^2 - 5xy$ 14. $x(p+q) + y(p+q)$

15. $ab(b-c) + bc(b-c) + ca(b-c)$

16. $ac + bc + ax + bx$ 17. $ax + ay - 4x - 4y$

18. $ab + a + bc + c$ 19. $ab + cd + bc + ad$

20. $ab + xy + by + xa$ 21. $3a^2 + 5bc - 5ab - 3ac$

22. $4x - ay^2 + 4y - axy$ 23. $1 + a + b + ab$

24. $a^x b^x + b^{2x} + b^{x+y}$

25. Evaluate by using a suitable formula:

(i) 35 x 36 + 35 x 4 : (ii) 47 x 21 – 47 x 16:

(iii) 58 x 17 + 17 x 42 : (iv) 3.69 x 63 + 3.69 x 37

8.5 FORMULA II

$$(a+b)^2 = a^2 + 2ab + b^2 \quad \forall\, a, b$$

Algebraic Proof

Formula II

$$(a+b)^2 = (a+b)(a+b) \quad \forall\, a, b$$

$$= a(a+b) + b(a+b)$$

(Distributivity of multiplication over addition)

$$= a^2 + ab + ab + b^2$$

$$= a^2 + 2ab + b^2$$

Formula III can also be proved in a similar way.

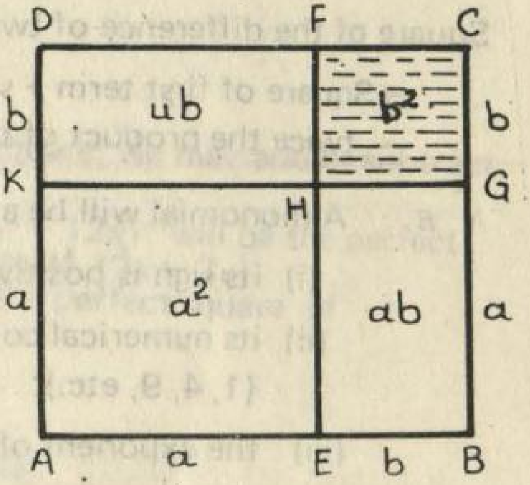
Geometrical Proof Of Formula II

In the adjoining figure,

$m\overline{AB} = m\overline{BC} = a + b$

$m\overline{AE} = m\overline{EH} = a$

$m\overline{EB} = m\overline{FC} = b$



Area of the square region ABCD = Area of the square region AEHK + area of the square region HGCF + area of the rectangular region EBGH + area of the rectangular region KHFD.

i.e. $(a + b)^2 = a^2 + b^2 + ab + ab$

$= a^2 + 2ab + b^2$

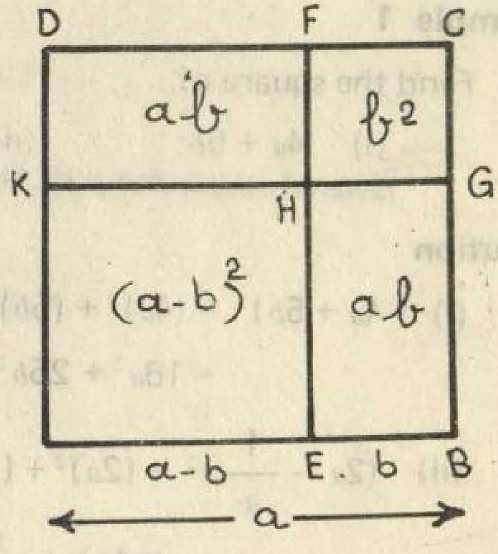
Geometrical Proof of Formula III

In the adjoining fiture,

$m\overline{AB} = m\overline{BC} = a$

$m\overline{EB} = m\overline{FC} = b$

$m\overline{AE} = m\overline{AK} = a - b$



Area of the square region AEHK = area of the square region ABCD — area of the rectangular region EBCF — area of the rectangular region KGCD + area of the square region HGCF.

i.e. $(a - b)^2 = a^2 - ab - ab + b^2$

$= c^2 - 2ab + b^2$

Formulae II and III may be expressed in words as follows:

Square of the sum of two terms

= Square of first term + square of second term

+ twice the product of terms.

Square of the difference of two terms

= Square of first term + square of second term

– twice the product of terms.

N. B.: A monomial will be a perfect square only if

(i) its sign is positive ;

(ii) its numerical co-efficient is a perfect square (1, 4, 9, etc.);

(iii) the exponent of each variable is an even number.

For instance, $9x^2$ is a perfect square, but $-9x^2$ is not a perfect square (its sign is negative) and $8x^2$ is also not a perfect square (its numerical co-efficient '8' is not a perfect square.) and $16x^3$ is also not a perfect square (because exponent 3 is not an even number)

Example 1

Find the square of

(i) $4a + 5b$; (ii) $2a - \frac{1}{a}$

Solution

(i) $(4a + 5b)^2 = (4a)^2 + (5b)^2 + 2(4a)(5b)$

$= 16a^2 + 25b^2 + 40ab$

(ii) $(2a - \frac{1}{a})^2 = (2a)^2 + (\frac{1}{a})^2 - 2(2a)(\frac{1}{a})$

$= 4a^2 + \frac{1}{a^2} - 4$

Example 2

What should we add or subtract for making a perfect square ?

(i) $4x^2 + 9y^2$ (ii) $9a^2 - 24ab$

Solution

(i) $4x^2 + 9y^2 = (2x)^2 + (3y)^2$

Therefore, for getting a perfect square, we may add or subtract

$2(2x)(3y) = 12xy$, $4x^2 + 9y^2 + 12xy$ will be the perfect square of $(2x + 3y)$

$(4x^2 + 9y^2) - 12xy$ will be the perfect square of

$$(2x - 3y)$$

(ii) $9a^2 - 24ab = (3a)^2 - 2(3a)(4b)$

i.e. Square of the first term minus twice the product of terms are there.

Therefore, for getting a perfect square, we have to add the square of the second term i.e. $(4b)^2 = 16b^2$

EXERCISE NO. 8.3

1. Fill in the empty frames to get true sentences (answer orally) :

(i) $(3x + 5y)^2 = \triangle^2 + 2\triangle\square + \square^2$

(ii) $(3x - 4y)^2 = \triangle^2 - 2\triangle\square + \square^2$

(iii) $(ab - c)^2 = a^2b^2 - 2\triangle c + \square^2$

(iv) $(3x - \square)^2 = (3x)^2 - 2\triangle y + \square^2$

(v) $(m - \frac{1}{n})^2 = m^2 - 2\triangle\square + \frac{1}{n^2}$

Find the square of the given expressions (questions No.2 to 10 should be done orally):

2. $p + q$ 3. $a + 4$ 4. $b - 5$

5. $6 - c$ 6. $2a + 3b$ 7. $3x - y$

8. $7 + 2y$ 9. $x - \frac{1}{x}$ 10. $y + \frac{1}{y}$

11. $4c + \frac{1}{2}$ 12. $2x - 3y$ 13. $2x - \frac{1}{x}$

14. $a - \frac{1}{2a}$ 15. $x^m + y^n$

What should be added or subtracted to make the given expression a perfect square ?

16. $a^2 + 9$ 17. $y^2 + \frac{1}{4}$ 18. $x^4 + 4$

19. $a^2 + 2ca$ 20. $4a^2 - 4ab$

8.6 USE OF FORMULAE II AND III

Example 1

Find the value of the following with the help of a suitable formula :

(i) 1003^2 (ii) 298^2

Solution

(i) $(1003)^2 = (1000 + 3)^2$

$$1000^2 + 3^2 + 2 \times 1000 \times 3$$

$$= 1000000 + 9 + 6000$$

$$= 1006009$$

(ii) $298^2 = (300 - 2)^2$

$$= 300^2 + 2^2 - 2 \times 300 \times 2$$

$$= 90000 + 4 - 1200$$

$$= 88804$$

Example 2

If $x - y = 7$, $xy = 8$, find the value of $x^2 + y^2$

Solution

$\because \quad x - y = 7$

$\therefore \quad (x - y)^2 = 7^2 = 49$

i.e. $\quad x^2 + y^2 - 2xy = 49$

or $\quad x^2 + y^2 - 2 \times 8 = 49$ (putting the value of xy)

i.e. $\quad x^2 + y^2 = 49 + 16 = 65$

Example 3

Evaluate $x^2 + \frac{1}{x^2}$ and $x^4 + \frac{1}{x^4}$ when $x - \frac{1}{x} = 4$

Solution

$\because \quad x - \frac{1}{x} = 4$

$\therefore \quad (x - \frac{1}{x})^2 = 4^2 = 16$

or $\quad x^2 + \frac{1}{x^2} + 2x . \frac{1}{x} = 16$

or $\quad x^2 + \frac{1}{x^2} - 2 = 16$

or $\quad x^2 + \frac{1}{x^2} = 16 + 2 = 18$

Now $\quad x^2 + \frac{1}{x^2} = 18$

$\therefore \quad (x^2 + \frac{1}{x^2})^2 = 18^2 = 324$

or $\quad x^4 + \frac{1}{x^4} + 2 . x^2 . \frac{1}{x^2} = 324$

or $\quad x^4 + \frac{1}{x^4} + 2 = 324$ or $x^4 + \frac{1}{x^4} = 324 - 2 = 322$

EXERCISE NO. 8.4

Evaluate by usong a suitable formula :

1. $(101)^2$ 2. $(403)^2$ 3. $(495)^2$ 4. $(688)^2$

Find the value of a^2+b^2, when

5. $a+b=5,\ ab=6$ 6. $a-b=7,\ ab=18$

7. $a+b=6,\ ab=-16$ 8. $a-b=1,\ ab=20$

Find the value of x^2+y^2, when

9. $x+y=14,\ xy=40$ 10. $x-y=-4,\ xy=21$

Find the value of $x^2+\frac{1}{x^2}$, when

11. $x+\frac{1}{x}=5$ 12. $x-\frac{1}{x}=7$

13. $x+\frac{1}{x}=3$ 14. $x-\frac{1}{x}=m$

Find the value of $x^4+\frac{1}{x^4}$, when

15. $x+\frac{1}{x}=4$ 16. $x-\frac{1}{x}=1$.

17. Find the value of

$\frac{a^2}{b^2}+\frac{b^2}{a^2}$ when $\frac{a}{b}+\frac{b}{a}=6$

18. if $\frac{x}{y}-\frac{y}{x}=3$, show that $\frac{x^2}{y^2}+\frac{y^2}{x^2}=11$

19. Find the value of

$4a^2+\frac{1}{a^2}$ when $2a-\frac{1}{a}=4$

20. Find the value of

$x^2+\frac{1}{4x^2}$ when $x-\frac{1}{2x}=5$

8.7 FACTORISATION OF EXPRESSIONS OF THE FORM $a^2 + 2ab + b^2$

We know that factorisation is the inverse process of multiplication . For instance, we have shown by multiplication that

$$(a+b)^2 = a^2 + 2ab + b^2 \quad \forall\, a, b \quad \ldots \quad (I)$$

By reversing the steps of multiplication we can show that the factorisation of $a^2 + 2ab + b^2$ is $(a+b)^2$. The two processes are compared below:

Multiplication		Factorisation	
$(a+b)^2$		$a^2 + 2ab + b^2$	(4)
$= (a+b)(a+b)$	... 1	$= a^2 + ab + ab + b^2$	... (3)
$= a(a+b) + b(a+b)$	... 2	$= a(a+b) + b(a+b)$	... (2)
$= a^2 + ab + ab + b^2$	... 3	$= (a+b)(a+b)$	... (1)
$= a^2 + 2ab + b^2$	... 4	$= (a+b)^2$	...

It may be noted that the four steps of factorisation can be obtained from the steps of multiplication by taking them in the reverse order, i.e., from the fourth, third, second and first steps respectively.

Similarly it can be shown that

$$a^2 - 2ab + b^2 = (a-b)^2 \quad \forall\, a, b \quad \ldots \quad (II)$$

From the relations (I) and (II) above it is clear that

(i) The first term (a^2) on the left hand side is the square of the first term (a) of the binomial on the right hand side ;

(ii) The third term (b^2) on the left hand side is the square of the second term ($\pm b$) of the binomial on the right hand side ; and

(iii) The middle term on the left hand side ($\pm 2ab$) is twice the product of the two terms ($a, \pm b$) of the binomial on the right hand side.

Example 1

Factorise $9x^2 - 12xy + 4y^2$

Solution

$$\begin{aligned}\text{Expression} &= 9x^2 - 12xy + 4y^2\\ &= (3x)^2 - 2(3x)(2y) + (2y)^2\\ &= (3x - 2y)^2\end{aligned}$$

Example 2

Evaluate with the help of a suitable formula :

$$1.79 \times 1.79 + 2 \times 1.79 \times 1.21 + 1.21 \times 1.21$$

Solution

Given expression

$$\begin{aligned}&= (1.79)^2 + 2(1.79)(1.21) + (1.21)^2\\ &= (1.79 + 1.21)^2\\ &= (3.00)^2 = 9\end{aligned}$$

EXERCISE NO: 8.5

Factorise (first five questions to be answered orally):

1. $x^2 + 2xy + y^2$
2. $p^2 - 2pq + q^2$
3. $4x^2 + 4xy + y^2$
4. $9x^2 - 12xy + 4y^2$
5. $25a^2 - 20ab + 4b^2$
6. $25 - 10c + c^2$
7. $x^2 + 8x + 16$
8. $36x^2 + 60xy + 25y^2$
9. $a^4 - 6a^2b^2 + 9b^4$
10. $25x^4y^4 - 10x^2y^2 + 1$
11. $x^{2n} + 2x^ny^n + y^{2n}$

Evaluate with the help of a formula :

12. $(6.8)^2 + 2\times 6.8 \times 3.2 + (3.2)^2$

13. $(535)^2 - 2 \times 535 \times 485 + (485)^2$

14. $(3.65)^2 + 2\times 3.65 \times 2.35 + (2.35)^2$

8.8 FORMULA IV

$$(a+b)(a-b) = a^2 - b^2 \quad \forall_{a,\,b}$$

Algebraic Proof

$$(a+b)(a-b) = a(a-b)+b(a-b) \quad \forall_{a,\,b}$$

(Distributivity of multiplication over addition)

$$= a^2 - ab + ba - b^2$$

$$= a^2 - ab + ab - b^2$$

(Commutative property of multiplication)

$$= a^2 - b^2$$

$$\therefore \quad (a+b)(a-b) = a^2 - b^2 \quad \forall_{a,\,b}$$

If we multiply $a+b$ and $a-b$ by the vertical method, we get the same result.

Geometrical Proof

In the adjoining figure,

$m\overline{AB} = m\overline{BC} = a$

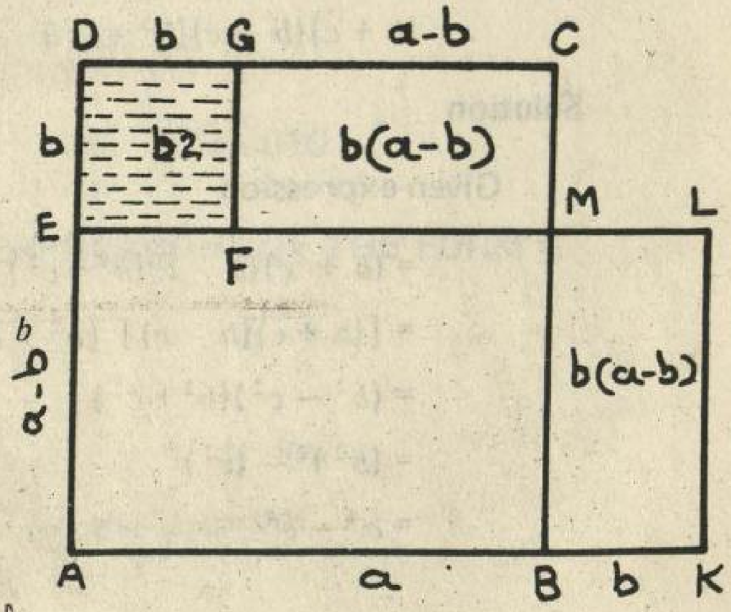
$m\overline{AK} = a + b$

$m\overline{EF} = m\overline{BK} = b$

$m\overline{AE} = m\overline{CG} = m\overline{BM} = a - b$

Area of the ractangular region AKLE

= area of the rectangular region ABME + area of the rectangular region BKLM



= area of the rectangular region ABME + area of the rectan lar region MCGF

= area of the square region ABCD − area of the square region EFGD

Thus $(a+b)(a-b) = a^2 - b^2$

This formula may be expressed like this :

"The product of the sum and difference of two numbers equals the difference of their squares."

Example 1

Use the formula to simplify $(x+3y)(x-3y)$

Solution

Given expression

$$= (x+3y)(x-3y)$$
$$= x^2 - (3y)^2$$
$$= x^2 - 9y^2$$

Example 2

Use the formula to find the product :

$$(b+c)(b-c)(b^2+c^2)$$

Solution

Given expression

$$= (b+c)(b-c)(b^2+c^2)$$
$$= [(b+c)(b-c)]\,(b^2+c^2)$$
$$= (b^2-c^2)(b^2+c^2)$$
$$= (b^2)^2 - (c^2)^2$$
$$= b^4 - c^4$$

EXERCISE NO. 8.6

Use the formula to simplify (first ten questions to be answered orally):

1. $(x+5)(x-5)$
2. $(x+a)(x-a)$
3. $(c+d)(c-d)$
4. $(x+\frac{1}{x})(x-\frac{1}{x})$
5. $(2a+\frac{1}{2a})(2a-\frac{1}{2a})$
6. $(3+2a)(3-2a)$
7. $(a^2+b^2)(a^2-b^2)$
8. $(8+a^2)(8-a^2)$
9. $(x+\frac{3}{4})(x-\frac{3}{4})$
10. $(a+\frac{b}{c})(a-\frac{b}{c})$

With the help of the formula find the continued product :

11. $(m+n)(m-n)(m^2+n^2)$
12. $(2-x)(2+x)(4+x^2)$
13. $(x-y)(x+y)(x^2+y^2)(x^4+y^4)$
14. $(x+1)(x^2+1)(x^4+1)(x-1)$

Use an appropriate formula to simplify :

15. $(x^m+y^m)(x^m-y^m)$
16. $(a^x+b^y)(a^x-b^y)$
17. $(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})$
18. $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$
19. 1025×975
20. 590×610

8.9 FACTORISATION OF EXPRESSIONS OF THE FORM $a^2 - b^2$

We have seen that

$$(a+b)(a-b)=a^2-b^2 \quad \forall\ a, b$$

Therefore, $a+b$ and $a-b$ are the factors of a^2-b^2.

Example 1

Factorise $4x^2 - 9y^2$

Solution

$$4x^2 - 9y^2 = (2x)^2 - (3y)^2$$

$$= (2x + 3y)(2x - 3y)$$

Example 2

Factrorise $2x^2 - 8y^2$

Solution

Neither of the term $2x^2$ nor $8y^2$ is a perfect square. However, 2 is common in the two terms and the remaining factor is a difference of two squares.

Thus given expression

$$= 2x^2 - 8y^2$$

$$= 2(x^2 - 4y^2)$$

$$= 2[x^2 - 2y)^2]$$

$$= 2(x + 2y)(x - 2y)$$

N.B. (i) Wherever possible take out the common factor before factorisation.

The operation of factorisation is said to be complete when no factor can be further factorised.

The following examples will clarify the point:

Example 3

Factorise $2a^5b - 32ab^5$

Solution

Expression

$$= 2a^5b - 32ab^5$$

$$= 2ab(a^4 - 16b^4)$$

$$= 2ab[(a^2)^2 - (4b^2)^2]$$

$$= 2ab(a^2 + 4b^2)(a^2 - 4b^2)$$

$$= 2ab(a^2 + 4b^2)[a^2 - (2b)^2]$$

$$= 2ab(a^2 + 4b^2)(a + 2b)(a - 2b)$$

Example 4

Evaluate with the help of the formula :

$$\frac{2.88 \times 2.88 - 1.12 \times 1.12}{2.88 - 1.12}$$

Solution

Given expression

$$= \frac{(2.88)^2 - (1.12)^2}{2.88 - 1.12}$$

$$= \frac{(2.88 + 1.12)(2.88 - 1.12)}{2.88 - 1.12}$$

$$= 2.88 + 1.12 = 4.00 = 4$$

EXERCISE NO. 8.7

Factorise (first ten questions should be answered orally):

1. $p^2 - q^2$ 2. $a^2 - (2b)^2$

3. $a^2 - 4$ 4. $x^2 - 9$

5. $81a^2 - 25b^2$ 6. $a^2x^2 - 4$ 7. $1 - x^2y^2$

8. $a^2b^2 - c^2$ 9. $x^2 - \frac{1}{4}$ 10. $a^2 - \frac{1}{a^2}$

11. $a^3 - ab^2$ 12. $x^3y - xy^3$ 13. $27 - 3a^2b^2$

14. $a^4 - b^4$ 15. $a^4 - 64$ 16. $3x^2 - 48$

17. $16x - x$ 18. $(x + y)^2 - z^2$ 19. $(a - b)^2 - c^2$

20. Simplify by using appropriate formula :

(i) $635 \times 635 - 365 \times 365$

(ii) $\dfrac{.599 \times .599 - .401 \times .401}{.599 - .401}$

(iii) $\dfrac{2.36 \times 2.36 - .04 \times .04}{2.36 + .04}$

(iv) $\dfrac{4.67 - 3.33}{4.67 \times 4.67 - 3.33 \times 3.33}$

8.10 FACTORISATION OF EXPRESSIONS REDUCIBLE TO THE FORM $a^2 - b^2$

Sometimes we have to factorise the expressions which do not appear to be of the form $a^2 - b^2$ but they can be converted into this form.

Example 1

Factorise $a^2 - c^2 - 2ab + b^2$

[ANALYSIS : First consider the problem in the light of the methods of factorisation studied so far :

(i) All the terms of the given expression do not contain a common factor.

(ii) It is not possible to partition the expression into two expressions, each containing a common factor.

(iii) It is not a trinomial which may be expressed as a complete square.

(iv) We have, therefore, to examine the fourth possibility of expressing it as a difference of two squares. The expression contains three squares. The fourth term is $-2ab$

which is twice the product of a and $-b$, $a^2 - 2ab + b^2$ is, therefore, a complete square. In view of this we proceed as follows].

Solution

Given expression

$$= a^2 - c^2 - 2ab + b^2$$

$$= (a^2 - 2ab + b^2) - c^2$$

$$= (a - b)^2 - c^2$$

$$= (a - b + c)(a - b - c)$$

Example 2

Factorise

$$4x^2 - y^2 - z^2 + 2yz$$

Solution

Expression

$$= 4x^2 - y^2 - z^2 + 2yz$$

$$= 4x^2 - (y^2 + z^2 - 2yz)$$

$$= (2x)^2 - (y - z)^2$$

$$= (2x + y - z)\,(2x - y + z)$$

Example 2

Factorise $a^4 + 4$

[**ANALYSIS : The given expression is not in the form of a difference of two squares. To convert it into that form we add and subtract $4a^2$ which is twice the product of a^2 and 2**

Solution

Given expression

$$= a^4 + 4 = (a^2)^2 + 2^2 + 2a^2.2 - 4a^2$$

$$(a^2 + 2)^2 - (2a)^2$$

$$= (a^2 + 2 + 2a)(a^2 + 2 - 2a)$$

$$(a^2 + 2a + 2)(a^2 - 2a + 2)$$

Example 4

Factorise $x^4 + x^2 + 25$

[ANALYSIS : The given expression contains two squares $x^4 = (x^2)^2$, $25 = 5^2$. Twice the product of x^2 and 5 is $10x^2$. Now both $x^4 + 10x^2 + 25$ and $x^4 - 10x^2 + 25$ are perfect squares. The third term of the given expression is x^2. In order to bring the expression in the form '$a^2 - b^2$' we should either :

(i) add $9x^2$ to $x^4 + x^2 + 25$

or

(ii) subtract $11x^2$ from $x^4 + x^2 + 25$

The first step will convert the whole expression into a difference of two squares whereas the second step will not do so. Therefore, we adopt the following procedure].

Solution

Given expression

$$= x^4 + x^2 + 25$$

$$= x^4 + x^2 + 25 + 9x^2 - 9x^2$$

$$= x^4 + 10x^2 + 25 - 9x^2$$

$$= (x^2 + 5)^2 - (3x)^2$$

$$= (x^2 + 5 + 3x)(x^2 + 5 - 3x)$$

$$= (x^2 + 3x + 5)(x^2 - 3x + 5)$$

Example 5

Factorise

$$a^4 - 11a^2b^2 + b^4$$

Solution

Given expression

$$\begin{aligned}
&= a^4 - 11a^2b^2 + b^4 \\
&= a^4 - 2a^2b^2 + b^4 - 9a^2b^2 \\
&= (a^2 - b^2)^2 - (3ab)^2 \\
&= (a^2 - b^2 + 3ab)(a^2 - b^2 - 3ab) \\
&= (a^2 + 3ab - b^2)(a^2 - 3ab - b^2)
\end{aligned}$$

N. B. In this example we have split up $-11a^2b^2$ into two parts $-2a^2b^2$, $-9a^2b^2$. Why ? Could we have split it up into the parts $-13a^2b^2$, $2a^2b^2$?

EXERCISE NO. 8.8

Factorise :

1. $a^2 + 2ab + b^2 - c^2$
2. $x^2 - 2xy + y^2 - z^2$
3. $a^2 - 2a + 1 - b^2$
4. $a^2 - 2ax - b^2 + x^2$
5. $c^2 + 2bc - a^2 + b^2$
6. $1 - a^2 - b^2 + 2ab$
7. $b^2 - c^2 - a^2 - 2ca$
8. $4a^2 - b^2 + 12ax + 9x^2$
9. $x^4 + 64$
10. $4x^4 + 81$
11. $x^4 + 4y^4$
12. $ax^4 + 4a$
13. $12a^4 + 3b^4$
14. $a^4 + a^2b^2 + b^4$
15. $x^4 + x^2y^2 + y^4$
16. $a^4 + a^2 + 1$
17. $x^4 + 2x^2 + 9$
18. $x^4 - 6x^2y^2 + y^4$
19. $a^4 - 7a^2b^2 + 9b^4$
20. $a^4 - 27a^2 + 1$

8.11 FORMULA V

$$(x + a)(x + b) = x^2 + (a + b)x + ab \ldots, a.b$$

Algebraic Proof

$$(x + a)(x + b) = x(x + b) + a(x + b) \quad \forall\, x, a, b$$

(D.M.A.)*

$$= x^2 + xb + ax + ab$$

(D.M.A.)

$$= x^2 + bx + ax + ab$$

(Commutativity of multiplication)

$$= x^2 + ax + bx + ab$$

(Commutativity of addition)

$$= x^2 + (a + b)x + ab$$

(D.M.A.)

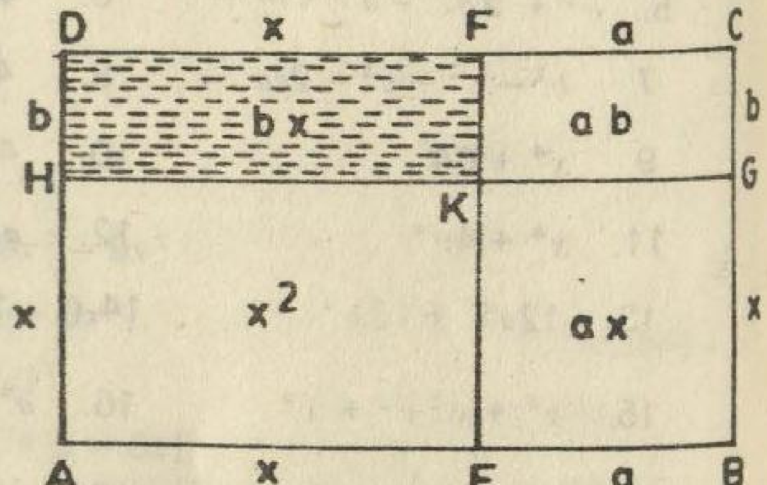
Geometrical Proof

In the following figure,

$m\overline{AE} = m\overline{AH} = m\overline{BG} = x$

$m\overline{EB} = m\overline{CF} = a$

$m\overline{GC} = m\overline{HD} = b$



Area of the rectangular region ABCD.

= Area of the square region AEKH

+ area of the rectangular region EBGK

+ area of the rectangular region HKFD

+ area of the rectangular region KGCF.

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* D.M.A. means distributivity of multiplication over addition.

i.e. $(x + a)(x + b) = x^2 + ax + bx + ab$

$$= x^2 + (a + b)x + ab$$

N.B. (a) Notice that x is the first term in each of the factors of $(x + a)$ $(x + b)$. The remaining terms are different.

(b) In the product :

(i) First term is the square of x, the first term in each of the factors $x + a$, $x + b$;

(ii) in the middle term $(a + b)x$ the co-efficient, $a + b$ is the sum of the unequal terms in the factors;

(iii) the third term (ab) is the product of the unequal terms.

Example 1

Find the product :

(i) $(x + 5)(x + 3)$; (ii) $(x - 4)(x - 6)$

Solution

(i) $(x + 5)(x + 3) = x^2 + (5 + 3)x + 5 \times 3$

$$= x^2 + 8x + 15$$

(ii) $(x - 4)(x - 6) = x^2 + (-4 - 6)x + (-4)(-6)$

$$= x^2 - 10x + 24$$

Example 2

Find the product :

(i) $(x + 5)(x - 3)$; (ii) $(x - 7)(x + 4)$

Solution

(i) $(x + 5)(x - 3) = x^2 + (5 - 3)x + 5 \times (-3)$

$$= x^2 + 2x - 15$$

(ii) $(x - 7)(x + 4) = x^2 + (-7 + 4)x + (-7) \times 4$

$$= x^2 - 3x - 28$$

EXERCISE NO. 8.9

Find the product with the help of formula (first five questions should be answered orally):

1. $(x+2)(x+3)$ 2. $(x+4)(x+5)$ 3. $(x-4)(x-3)$

4. $(x-1)(x-5)$ 5. $(x+4)(x-1)$ 6. $(x+12)(x-5)$

7. $(x-15)(x+12)$ 8. $(x-18)(x+3)$

9. $(x-5)(x+1)$ 10. $(x-6)(x-5)$

8.12 FACTORISATION OF EXPRESSIONS OF THE FORM

$$x^2+(a+b)x+ab = x^2+ax+bx+ab \quad \forall a,\ b \quad \text{(D.M.A.)}$$

$$= x(x+a)+b(x+a) \quad \text{(D.M.A.)}$$

$$= (x+a)(x+b) \quad \text{(D.M.A.)}$$

It should be noted that the third term of the given expression has the factors a, b whose algebraic sum $a+b$ is the co-efficient of the middle term involving x. The middle term has been split up into the terms ax, bx whose co-efficients, a, b are the factors of ab. This facilitates factorisation. The following examples illustrate the method.

Example 1

Factorise $x^2+7x+12$

(The factors of 12 whose algebraic sum is the numerical co-efficient (+7) of the middle term are 3,4. Therefore, we write the middle term $+7x$ as $(3x+4x)$.

Solution

$$x^2+7x+12 = x^2+(3x+4x)+12$$

$$= (x^2+3x)+(4x+12)$$

$$= x(x+3)+4(x+3)$$

$$= (x+3)(x+4)$$

Example 2

Factorise $x^2 - 8x + 12$

(The factors of +12 whose sum is the numerical coefficient (−8) of the middle term are −6 and −2. Therefore, we write $-8x$ as $(-6x - 2x)$.

Solution

$$x^2 - 8x + 12 = x^2 - 6x - 2x + 12$$

$$= x(x-6) - 2(x-6)$$

$$= (x-6)(x-2)$$

Example 3

Factorise $x^2 - 10x - 24$

(First we find two factors of −24 in such a way that their algebraic sum is −10. Clearly one factor should be positive and the other negative. The suitable factors are −12 and + 2. Therefore, we write the middle term as $-12x + 2x$.

Solution

$$x^2 - 10x - 24 = x^2 - 12x + 2x - 24$$

$$= x(x-12) + 2(x-12)$$

$$= (x-12)(x+12)$$

Example 4

Factorise $a^2 + 13ab - 30b^2$

(The numerical co-efficient of the last term is −30. We want its two factors which are such that their algebraic sum is +13. Such factors are +15 and −2 Therefore, we split the middle term into two parts $+15ab$ and $-2ab$).

Solution

$$
\begin{aligned}
a^2 + 13ab - 30b^2 &= a^2 + 15ab - 2ab - 30b^2 \\
&= a(a + 15b) - 2b(a + 15b) \ \forall a, b \\
&= (a + 15b)(a - 2b)
\end{aligned}
$$

N.B. : From the above solved examples we conclude that :

(i) if the sign of the last term of the expression is positive, the middle term will be split up into parts having the same sign positive or negative according to the sign of the middle term (as in Examples 1 and 2).

(ii) if the last term is negative, the middle term will be split into two parts with opposite signs, the numerically greater part having the sign of the middle term (as in Examples 3 and 4).

EXERCISE NO. 8.10

Fill in the blank frames (Orally):

1. $x^2 + 9x + 20 = (x + \triangle)(x + \square)$
2. $x^2 - 5x + 4 = (x - \triangle)(x - \square)$
3. $x^2 + 10x - 24 = (x + \triangle)(x - \square)$
4. $x^2 - 4x - 12 = (x - \triangle)(x + \square)$

Factorise (question 5 to 10 to be answered orally) :

5. $x^2 + 7x + 10$ 6. $x^2 + 12x + 35$

7. $x^2 - 10x + 21$ 8. $x^2 - 8x + 7$

9. $x^2 + 2x - 99$ 10. $x^2 + 13x + 30$

11. $p^2 + 10pq + 24q^2$ 12. $x^2 - x - 6$

13. $x^2 - 17x + 60$ 14. $x^2 + 17x - 60$

15. $x^2 - 25x - 84$ 15. $x^2 + 15x - 54$

17. $2x^2 + 8x + 6$ 18. $3x^2 - 30x - 72$

19. $x^3y^3 - 9x^2y^2 + 20xy$ 20. $x^3 + 2x^2 - 24x$

8.13 FORMULA VI

$$\begin{aligned}(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \qquad \forall\, a, b\\ &= a^3 + b^3 + 3ab(a+b) \qquad \forall\, a, b\end{aligned}$$

Formula VII

$$\begin{aligned}(a-b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3\\ &= a^3 - b^3 - 3ab(a-b)\end{aligned}$$

Algebraic Proof

Formula VI

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2 \qquad \forall\, a, b\\ &= (a+b)(a^2 + 2ab + b^2)^*\\ &= a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2)\\ &= \qquad \text{(D.M.A.)}\\ &= a^3 + 2a^2b + ab^2 + ba^2 + 2ab^2 + b^3\\ &\qquad \text{(D.M.A.)}\\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

(Associative and commutative properties of addition)

Again $a^3 + 3a^2b + 3ab^2 + b^3$

$$\begin{aligned}&= a^3 + b^3 + 3a^2b + 3ab^2\\ &= a^3 + b^3 + 3ab(a+b)\end{aligned}$$

Formula VII may be proved in a similar way.

Example 1

Find the cube of $2x + 5y$

---

* These two factors may be multiplied by the usual vertical method also.

Solution

$$(2x + 5y)^3 = (2x)^3 + 3(2x)^2(5y) + 3(2x)(5y)^2 + (5y)^3$$
$$= 8x^3 + 3.4x^2 \,.\, 5y + 3.2x.25y^2 + 125y^3$$
$$= 8x^3 + 60x^2y + 150xy^2 + 125y^3$$

Example 2

Expand $(3x - 4)^3$

Solution

$$(3x - 4)^3 = (3x)^3 - 3(3x)^2 \,.\, 4 + 3(3x).4^2 - 4^3$$
$$= 27x^3 - 108x^2 + 144x - 64^*$$

Example 3

Find the value of $x^3 + y^3$ given that $x + y = 6$, $xy = 8$.

Solution

$\because \quad x + y = 6$

$\therefore \quad (x + y)^3 = 6^3$

or $\quad x^3 + y^3 + 3xy(x + y) = 216$

or $\quad x^3 + y^3 + 3.\,8.\,6 = 216$ [Putting values of xy and $(x + y)$]

or $\quad x^3 + y^3 + 144 = 216$

or $\quad x^3 + y^3 + 144 - 144 = 216 - 144$

$\therefore \quad x^3 + y^3 = 72$

Example 4

Find the value of $x^3 - \dfrac{1}{x^3}$ when $x - \dfrac{1}{x} = 4$

Solution

$\because \quad x - \dfrac{1}{x} = 4,$

---

* Notice that the terms occur in descending order w.r.t. the variable x and the signs of the terms are alternately positive and negative.

$$(x - \frac{1}{x})^3 = 4^3 = 64$$

or $x^3 - \frac{1}{x^3} - 3.x.\frac{1}{x}.(x - \frac{1}{x}) = 64$

or $x^3 - \frac{1}{x^3} - 3.4 = 64$

or $x^3 - \frac{1}{x^3} - 12 = 64$

or $x^3 - \frac{1}{x^3} - 12 + 12 = 64 + 12$

$\therefore$ $x^3 - \frac{1}{x^3} = 76$

EXERCISE NO. 8.11

Fill in the blanks to complete the sentences (orally):

1. $(a + 2b)^3 = (\ ....\ )^3 + 3a^2 . 2b + 3a.(2b)^2 + (.......)^3$
2. $(2x - y)^3 = (........\ )^3 - 3(......)^2 . y + 3(2x) . y^2 - y^3$
3. $(5 + x)^3 = 5^3 + 3\ (.......)^2 (.......) + 3(........)(.......)^2 + x^3$
4. $(3x - 2a)^3 = (3x)^3 - 3(.....)^2 (......) + 3(.....)(2a)^2 - (2a)^3$

Find the cube of

5. $m + n$ 6. $2 + c$ 7. $3a - 2b$

8. $4 - 2x$ 9. $x + \frac{1}{x}$ 10. $\frac{x}{y} - \frac{y}{x}$

Find the value of

11. $x^3 + y^3$ when $x + y = 5$, $xy = 6$
12. $x^3 + y^3$ when $x + y = 1$, $xy = 12$
13. $a^3 + b^3$ when $a + b = 2$, $ab = -8$
14. $x^3 - y^3$ when $x - y = 2$, $xy = 15$
15. $a^3 - b^3$ when $a - b = 3$, $ab = 28$
16. $x^3 + \frac{1}{x^3}$ when (i) $x + \frac{1}{x} = 4$, (ii) $x + \frac{1}{x} = 3$

17. $x^3 - \frac{1}{x^3}$ when (i) $x - \frac{1}{x} = 4$ (ii) $x - \frac{1}{x} = -4$

18. If $x + y = 2$, show that $x^3 + y^3 + 6xy = 8$

19. If $a - b = 3$, show that $a^3 - b^3 - 9ab = 27$

20. If $x = 2 - \frac{1}{x}$, prove

$$x^3 + \frac{1}{x^3} = x^2 + \frac{1}{x^2} = x^4 + \frac{1}{x^4}$$

8.14 FACRORISATION OF EXPRESSIONS OF THE FORM

$$a^3 \pm 3a^2b + 3ab^2 \pm b^3$$

We have seen that

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \qquad \forall\, a, b$$

Therefore,

$$(a^3 + 3a^2b + 3ab^2 + b^3) = (a+b)^3 \quad \forall\, a, b$$

Similarly,

$$a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3 \quad \forall\, a, b$$

Example 1

Express $x^3 + 6x^2 + 12x + 8$ as a cube of a binomial.

Solution

Given Expression

$= x^3 + 6x^2 + 12x + 8$ as a cube of a binomial.

$= x^3 + 3 . x^2 . 2 + 3 . x . 2^2 + 2^3$

$= (x + 2)^3$

Example 2

Factorise $8a^3 - 6a^2b + 150ab^2 - 125b^3$

Solution

$$
\begin{aligned}
& 8a^3 - 60a^2b + 150ab^2 - 125b^3 \\
= \quad & (2a)^3 - 3(2a)^2(5b) + 3(2a)(5b)^2 - (5b)^3 \\
= \quad & (2a - 5b)^3
\end{aligned}
$$

EXERCISE NO. 8.12

Express in the form of a cube (orally):

1. $x^3 + 3x^2 . 4y + 3.x.(4y)^2 + (4y)^3$
2. $(2a)^3 - 3(2a)^2(3b) + 3(2a)(3b)^2 - (3b)^3$
3. $x^3 - 3x^2 .3 + 3.x.3^2 - 3^3$

Factorise :

4. $x^3 + 3x^2y + 3xy^2 + y^3$
5. $p^3 + 9p^2 + 27p + 27$
6. $x^3 - 15x^2y + 75xy^2 - 125y^3$
7. $1 - 3x + 3x^2 - x^3$
8. $x^3 + 3x + \dfrac{3}{x} + \dfrac{1}{x^3}$
9. $m^3 - 3m + \dfrac{3}{m} - \dfrac{1}{m^3}$
10. $1000 - 300xy + 30x^2y^2 - x^3y^3$

8.15 FORMULA VIII

$$(a + b)(a^2 - ab + b^2) = a^3 + b^3 \qquad \forall\ a, b$$

Proof of Formula VIII

$$(a + b)(a^2 - ab + b^2) = a(a^2 - ab + b^2) + b(a^2 - ab + b^2)$$

(D.M.A.) $\qquad \forall a, b$

$$
\begin{aligned}
&= a^3 - a^2b + ab^2 + ba^2 - ab^2 + b^3 \\
&= a^3 - a^2b + a^2b + ab^2 - ab^2 + b^3 \\
&= a^3 + b^3
\end{aligned}
$$

[associative and commutative properties of addition]

Formula IX may be proved similarly

N.B. : (i) Notice that in both the formulae the sign between the two terms of the first factor on the left hand side agrees with the sign between the cubes on the right hand side.

(ii) The sign before the product term ab in the second term is opposite to that between a. b in the first factor.

Example 1

Simplify with the help of formula

$$(2a - 5b)(4a^2 + 10ab + 25b^2)$$

Solution

$$(2a - 5b)(4a^2 + 10ab + 25b^2)$$
$$= (2a - 5b)[(2a)^2 + (2a)(5b) + (5b)^2]$$
$$= (2a)^3 - (5b)^3$$
$$= 8a^3 - 125b^3$$

Example 2

Find the continued product with the help of formulae

$$(1 + x)(1 - x)(1 + x + x^2)(1 - x + x^2) .$$

Solution

$$(1 + x)(1 - x)(1 + x + x^2)(1 - x + x^2)$$
$$= [(1 + x)(1 - x + x^2)]\,[(1 - x)(1 + x + x^2)]$$
$$= [(1 + x)(1^2 - 1.\,x + x^2)]\,[(1 - x)(1^2 + 1.\,x + x^2)]$$
$$= (1^3 + x^3)(1^3 - x^3)$$
$$=(1^3)^2 - (x^3)^2 = 1 - x^6$$

EXERCISE NO: 8.13

Simplify with the help of formulae (answer first five questions orally):

1. $(c+d)(c^2-cd+d^2)$
2. $(l+m)(l^2-lm+m^2)$
3. $(x-y)(x^2+xy+y^2)$
4. $(p-q)(p^2+pq+q^2)$
5. $(x+3)(x^2-3x+9)$
6. $(2a-3)(4a^2+6a+9)$
7. $(a^2-1)(a^4+a^2+1)$
8. $(x+\frac{1}{x})(x^2-1+\frac{1}{x^2})$
9. $(\frac{x}{y}-\frac{y}{x})(\frac{x^2}{y^2}+1+\frac{y^2}{x^2})$
10. $(x^n+y^n)(x^{2n}-x^ny^n+y^{2n})$
11. $(a^m-y)(a^{2m}+a^my+y^2)$

Using formulae find the continued products :

12. $(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$
13. $(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)$
14. $(a+3)(a-3)(a^2-3a+9)(a^2+3a+9)$
15. $(3a+2b)(3a-2b)(9a^2+6ab+4b^2)(9a^2-6ab+4b^2)$

8.16 FACTORISATION OF EXPRESSIONS OF THE FORM

$a^3 \pm b^3$

We know that

$$(a+b)^3 = a^3+b^3+3ab(a+b) \quad \forall_{a,\,b}$$

$$a^3+b^3+3ab(a+b) = (a+b)^3$$

or $a^3+b^3 = (a+b)^3 - 3ab(a+b)$

(Subtraction property of an equation)

$$= (a+b)[(a+b)^2-3ab]$$

D.M.A.)

$$= (a+b)[a^2+b^2+2ab-3ab]$$

$$= (a+b)(a^2 - ab + b^2)$$

Thus $\quad a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Similarly, we can show that

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Example 1

Factorise $\quad 8x^3 + 27$

Solution

$$\begin{aligned} 8x^3 + 27 &= (2x)^3 + 3^3 \\ &= (2x+3)(2x)^2 - (2x)(3) + 3^2] \\ &= (2x+3)(4x^2 - 6x + 9) \end{aligned}$$

Example 2

Factorise $\quad x^6 - 64$

Solution

$$\begin{aligned} x^6 - 64 &= (x^3)^2 - (8)^2 \\ &= (x^3 + 8)(x^3 - 8) \\ &= (x^3 + 2^3)(x^3 - 2^3) \\ &= (x+2)(x^2 - 2x + 4)(x-2)(x^2 + 2x + 4) \\ &= (x+2)(x-2)(x^2 - 2x + 4)(x^2 + 2x + 4) \end{aligned}$$

Alternative Method

$$\begin{aligned} x^6 - 64 &= (x^2)^3 - 4^3 \\ &= (x^2 - 4)([(x^2)^2 + 4x^2 + 4^2)] \\ &= (x^2 - 4)(x^4 + 4x^2 + 16) \\ &= (x+2)(x-2)[(x^2)^2 + 4^2 + 2.x^2.4 - 4x^2] \\ &= (x+2)(x-2)[(x^2+4)^2 - (2x)^2] \\ &= (x+2)(x-2)(x^2 + 4 + 2x)(x^2 + 4 - 2x) \\ &= (x+2)(x-2)(x^2 + 2x + 4)(x^2 - 2x + 4) \end{aligned}$$

Example 3

Simplify with the help of formula

$$\frac{(3.57)^3 + (2.43)^3}{(3.57)^2 - 3.57 \times 2.43 + (2.43)^2}$$

Solution

Let 3.57 = a, 2.43 = b

Given expression

$$= \frac{a^3 + b^3}{a^2 - ab + b^2}$$

$$= \frac{(a+b)(a^2 - ab + b^2)}{a^2 - ab + b^2}$$

$$= (a+b)$$

$= 3.57 + 2.43$ (Replacing values of a, b)

$= 6.00 = 6$

EXERCISE NO. 8.14

Factorise (first five questions to be answered orally):

1. $x^3 + y^3$ 2. $x^3 + 64$ 3. $a^3 + 8$
4. $x^3 - z^3$ 5. $x^3 - 125$ 6. $27x^3 + 64$
7. $x^3 - y^3z^3$ 8. $a^4b - ab^4$ 9. $a^2b^5 - a^5b^2$
10. $x^6 + y^6$ 11. $x^6 + 64y^6$ 12. $x^6 - y^6$
13. $x^6 - 1$ 14. $a^7b - ab^7$ 15. $x^3y^6 - 64$
16. $a^9 - b^9$ 17. $a^6 - 729$

Evaluate with the help of a formula :

18. $$\frac{(416)^3 + (84)^3}{(416)^2 - 416 \times 84 + (84)^2}$$

19. $$\frac{(652)^3 - (252)^3}{(652)^2 + 652 \times 252 + (252)^2}$$

20. $$\frac{(4.49)^3 + 3.51 \times 3.51 \times 3.51}{4.49 \times 4.49 - 3.51 \times 4.49 + (3.51)^2}$$

MISCELLANEOUS EXERCISE NO. 8.15

Find the value of :

1. $x^2 + y^2$ when $x - y = 2, \ xy = 8$
2. $4d^2 + \frac{1}{d^2}$ when $2d - \frac{1}{d} = 3;$
3. $x^4 + \frac{1}{x^4}$ when $x - \frac{1}{x} = 2 ;$
4. $a^3 + b^3$ when $a + b = 5, \ ab = -6$
5. $x^3 - y^3$ when $x - y = 3, \ xy = 10 ;$
6. $x^3 - \frac{1}{x^3}$ when $x - \frac{1}{x} = 4 .$

Simplify by using formulae :

7. (i) 1005^2 (ii) 698^2.
8. $(7.8)^2 - 2 \times 3.9 \times 2 \times 4.8 + (4.8)^2$
9. $$\frac{.321 \times .321 - .179 \times .179}{.321 - .179}$$
10. $$\frac{45 \times 45 + 2 \times 35 \times 45 + (35)^2}{45 \times 45 - 35 \times 35}$$
11. $(3.75)^3 + 3(3.75)^2 \times 1.25 + 3 \times 3.75 \times (1.25)^2 + (1.25)^3$
12. $$\frac{.89 \times .89 \times .89 - .64 \times .64 \times .64}{.89 \times .89 + .89 \times .64 + .64 \times .64}$$

Factorise :

13. $x^3 - x^2y + xy^2$ 14. $x^2 - xy - 3x + 3y$

15. $1 + a^2b^2 + a^2 + b^2$ 16. $(a + b)^2 + 3(a + b)$

17. $ab(x^2+y^2) - xy(a^2+b^2)$ 19. $(ax+by)^2+(bx-ay)^2$

19. $(ab+cd)^2+(ac-bd)^2$ 20. $25a^2b^2-20abc+4c^2$

21. $a^{2x}-2a^xb^y+b^{2y}$ 22. $81a^2 - 25$

23. a^3-9ab^2 24. $x^4 - 81$

25. $x^8 - y^8$ 26. $a^{4x} - b^{4x}$

27. $c^2 - 2bc - a^2 + b^2$ 28. $2xy - x^2 - y^2 + z^2$

29. $8ax + x^2 - b^2 + 16a^2$ 30. $4x^4 + 1$

31. $a^4 + 64$ 32. $a^4 + 14a^2b^2 + 81b^4$

33. x^4+17x^2+81 34. $x^4-3x^2y^2+y^4$

35. $64-25b^2+b^4$ 36. $a^3b^3-10a^2b^2+24ab$

37. $x^2 - x - 2$ 38. $a^2+ab-12b^2$

39. $a^3 + 27b^3$ 40. $x^4 - 27x$

41. $x^{3m} + y^{3m}$ 42. $64x^6 - 729$

43. x^2+2x^2-x-2 44. x^3-5x^2-x+5

45. x^4-x^3+8x-8

FORMULAE AT A GLAN

I. $k(a+b+c) = ka+kb+kc \quad \forall a, b, k$

II. $(a+b)^2 = a^2+2ab+b^2 \quad \forall a, b$

III. $(a-b)^2 = a^2-2ab+b^2 \quad \forall a, b$

IV. $(a+b)(a-b) = a^2 - b^2 \quad \forall a, b$

V. $(x+a)(x+b) = x^2+(a+b)x+ab \quad \forall x, a, b$

VI. $(a+b)^3 = a^3+3a^2b+3ab^2+b^3 \quad \forall a, b$

$= a^3+b^3+3ab(a+b)$

VII. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$= a^3 - b^3 - 3ab(a-b)$

VIII. $(a+b)(a^2 - ab + b^2) = a^3 + b^3 \quad \forall\, a, b$

IX. $(a-b)(a^2 + ab + b^2) = a^3 - b^3 \quad \forall\, a, b$

2. Procedure of factorisation

For factorising a given expression one or more of the following steps may be taken :

(a) Take out the common factor , if there be any.

(b) When the given expression is a binomial ;

- (i) See if it is a difference of two squares, if so, use formula IV;
- (ii) If it is a sum or difference of two cubes use formula VIII or IX, whichever is applicable.

(c) In case the given expression is a trinomial, there are three possible ways of factorisation.

- (i) The product of the co-efficients of x^2 and the constant term should be so factorised that the algebraic sum of the factors is equal to the co-efficient of the middle (x) term. The remaining process may be completed in the usual way ;
- (ii) See if the expression is a complete square. If so, use Formula II or III.
- (iii) If feasible, express the expression as a difference of two squares and use Formula IV.

(d) When the given expression has four terms:

- (i) See if it can be partitioned into two groups, each having a common factor.
- (ii) If possible, express the expression as a difference of two squares and use Formula IV.
- (iii) See if the expression can be expressed as a complete cube.

CHAPTER 9

ALGEBRAIC SENTENCES, MATRICES AND EQUATIONS

9.1 GRAPHIC REPRESENTATION OF A LINEAR EQUATION IN TWO VARIABLES AND ITS SOLUTION SET

You have become familiar with algebaric expressions in Chapter 7. Such expressions may involve one variable or more. Below are given a few examples of algebraic expressions involving two variables:

$$2x+3y,\ 5xy,\ ax+by+c$$

The last letters of alphabet x, y, etc. are generally used to denote variables whereas the letters from the beginning a, b, c, etc., denote constants. Algebraic expressions are used to make ALGEBRAIC SENTENCES. Some example of algebraic sentences are as follows:

$$3x+5y<2,\ 2x+3y>4,\ 5xy=10,\ ax+by+c=0$$

The three symbols used here, $<$. $>$. $=$, have respectively, the meanings 'is less than', 'is greater than', 'is equal to'. A sentence in which the symbol of equality (=) is used, is called an EQUATION. In the above examples the last two sentences, namely

$$5xy=10,\ ax+by+c=0$$

are equations.

A sentence may be TRUE or FLASE. Whether a sentence is true or false depends on the values given to the variables involved in the sentence. By solution of an equation we mean finding the values of the variable or variables involved in it, which make it a true sentence. The set of all such values is called the SOLUTION SET of the equation. As an example the solution set of the equation

$$2x + 3 = 7$$

which involves one variable x, is $\{2\}$, because only the number 2 satisfies this equation. The set $\{2\}$ is a sub-set of the set of real numbers, which we shall denote by R.

Similarly, the solution set of the equation $2x^2 = 8$ is $\{2, -2\}$. In general, the solution set of every equation in one variable is a sub-set of R. Similarly, the solution set of an equation in two variabies is a sub-set of R x R (Cartesian product of R with itself). It is assumed that every equation in one or two variables has real roots. In fact, a solution set of an equation in two variables is an ordered pair of the form (x, y) and it is clear that a set of such ordered pairs is a sub-set of R x R. For example, one solution of the equation.

$$x - y + 1 = 0$$

is the ordered pair (1, 2), i.e., the values of $x = 1$, $y = 2$, satisfy this equation. One examining the equation more closely, we find that the pair (1, 2) is not its only solution, but the following pairs are also solutions of the equation:

$$(-3, -2), (-2, -1), (-1, 0), (0, 1), (2, 3), (3, 4).$$

Not only this, there are numerous other ordered pairs which can be the members of the solution set of the equation, e.g., (−4, −3), (4, 5), etc.

Unless we find all the ordered pairs of this equation, we cannot be justified in writing down its solution set.

Because each solution of an equation in two variables is an ordered pair, one method of finding all such ordered paris consists in giving arbitrary values to one of the variables, say x one by one and finding the corresponding values of y from the given equation. The following table illustrates the method:

$$x - y + 1 = 0 \text{ or } y = x + 1$$

x	-10	$-9\frac{1}{2}$	-7.2	-1.4	0.2	1.8	2.1	3.3	4.5	6	6.9	10
y	-9	$-8\frac{1}{2}$	-6.2	-0.4	1.2	2.8	3.1	4.3	5.5	7	7.9	11

In this table we have given twelve values successively to x from -10 to 10, and obtaining the corresponding twelve values of y. We have written them beneath the values of x. For example, corresponding to the value -10 of x, we get -9 as the corresponding value of y. Thus we get $(-10, -9)$ as one solution of the equation.

Similarly, we have obtained eleven other ordered pairs each of which is a solution of the given equation. But the fact which becomes all the more evident is that these twelve ordered pairs along with those mentioned above do not exhaust the solutions of the equations. Besides them, there are other innumerable ordered pairs which satisfy the given equation, because whatever value we may give to x, we can always find a corresponding value of y. Since there is no restriction on the value of x, it can be given infinite number of values and the corresponding values of y can also be obtained. In this way we shall get innumerable ordered pairs which will be solutions of the equation $x - y + 1 = 0$.

This shows that the solution set of this equation is an infinite set.

EXERCISE No. 9.1.

Find any ten components (ordered pairs) of the solution set of each of the following equations. Are the solution sets of these equations infinite sets?

1. $x + y = 0$. 2. $x - y = 0$.

2. $2x + 3y = 6$. 4. $x = 2 + y$.

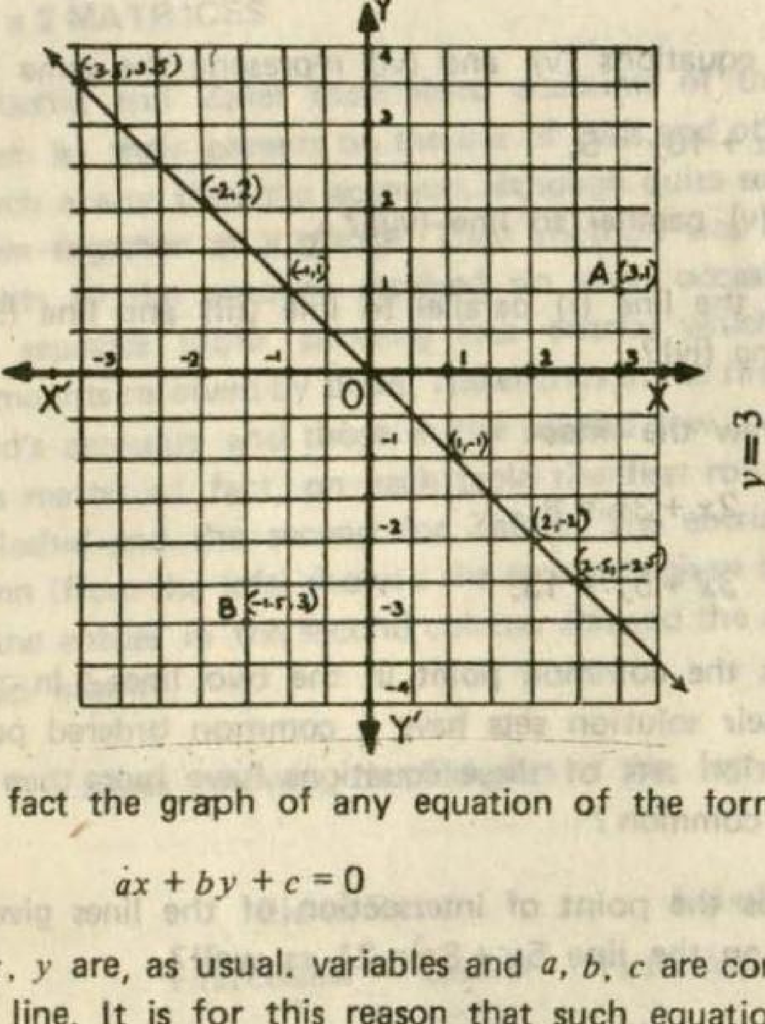
9.2. GRAPH OF A LINEAR EQUATION IN TWO VARIABLES

Graphical representation of a linear equation in two variables very clearly demonstrates the fact that its solution set is an infinite set. An ordered pair is represented on graph paper by a point whose position is specified with reference to perpendicular lines drawn on the paper. The elements (or components) of the ordered pair are represented by the distances of the point from the fixed lines, called axes. A number of ordered pairs, obtained from the given equation, in the form of a table (as in section 9.1 above), are represented by points marked on the graph paper. The points are joined by means of a line. To every point of this line there corresponds an ordered pair which is a solution of the given equation. In this way by drawing a graph of the equation we get a representation of its solution set. As a matter of fact it is the whole line which represents the solution set of the equation. (Remember that a line extends infinitely in both directions).

The following figure shows the solution set of the equation $x + y = 0$ in the form of a line.

$$x + y = 0 \quad \text{or} \quad x = -y$$

x	-2	-1	0	1	2
y	2	1	0	-1	-2



In fact the graph of any equation of the form

$$ax + by + c = 0$$

where x, y are, as usual, variables and a, b, c are constants, is a straight line. It is for this reason that such equation is called LINEAR EQUATION. Such equation is also called a FIRST DEGREE EQUATION in x, y.

EXERCISE No. 9.2

1. Draw the graphs of the equations given in Exercise No. 9.1.

(Since the graph of a linear equation is always a line, for the sake of brevity, we call such an equation a line).

2. Draw the graphs of the lines:

(i) $x = 0$, (ii) $y = 0$, (iii) $x = 3$,

(iv) $y = -5$, (v) $3x + 5y = 8$, (vi) $6x + 10y = 16$

Do the equations (v) and (vi) represent the same line?

(vii) $6x + 10y = 5$.

Is line (v) parallel to line (vii)?

(viii) Is the line (i) parallel to line (iii) and line (ii) parallel to line (iv)?

(ix) Draw the lines

$$2x + 3y = 8,$$

$$3x + 5y = 13.$$

Which is the common point in the two lines? In other words, do their solution sets have a common ordered pair? Can the solution sets of these equations have more than one ordered pair common?

(x) Does the point of intersection of the lines given in part (ix) lie on the line $5x + 8y = 21$ as well?

(xi) Is any ordered pair common to the solution sets of the equations

$$2x - 3y = 6,$$

$$4x - 6y = 5?$$

If not, why? Explain by drawing a graph.

(xii) Do the equations

$$3x + 5y = 9$$

$$5x - 2y = 9 + 2x - 7y$$

represent the same line ?

9.3 2 x 2 MATRICES

Rashid and Zafar maintained accounts of the gifts given to them by their parents on the eve of Eids and other occasions in such a way that the accounts although quite separate, could be seen together at a glance. Their method was like this. The accounts of the amount received on each occasion was kept on a separate table showing four entries which represented the amounts received by them. The entries in the first row showed Rashid's amounts and those in the second row showed Zafar's. (As a matter of fact, on each table the first row was reserved for Rashid and the second for Zafar). The entries in the first column (from the left) showed the amounts given by their father and the entries in the second column showed the amounts given by their mother.

The tables showing the Eid gifts of the last year were like this:

$$\begin{array}{lcc@{\qquad}cc}
 & \multicolumn{2}{c}{\text{Eid-ul-Fitr}} & \multicolumn{2}{c}{\text{Eid-ul-Azha}} \\
 & \text{First column} & \text{Second column} & \text{First column} & \text{Second column} \\
 & \text{(Father)} & \text{(Mother)} & \text{(Father)} & \text{(Mother)} \\
\text{First Row} \rightarrow \text{Rashid} & 5 & 4 & 4 & 3 \\
\text{Second Row} \rightarrow \text{Zafar} & 4 & 3 & 5 & 3
\end{array}$$

By examining these tables answer the following questions:

(i) What Eid amount did Rashid get from his father on Eid-ul-Fitr ?

(ii) What Eid amount did Zafar get from his mother on Eid-ul-Azha ?

(iii) What amount did each brother get on both the Eids taken together ?

(iv) Do the entries of the following table provide answers to question (iii) ?

$$\boxed{\begin{matrix} 9 & 7 \\ 9 & 6 \end{matrix}}$$

Since this table shows the amounts received by them (separately) from their parents on the two Eids taken together, it is called the 'sum' of first two tables and the process is shown like this:

$$\boxed{\begin{matrix} 5 & \textcircled{4} \\ 4 & 3 \end{matrix}} + \boxed{\begin{matrix} 4 & \textcircled{3} \\ 5 & 3 \end{matrix}} = \boxed{\begin{matrix} 9 & \textcircled{7} \\ 9 & 6 \end{matrix}}$$

By this time you must have got an idea of the method of finding the entries on the 'sum table'. Each entry on this table is the sum of the corresponding entries on the two 'Eid tables, e.g., the entry on the sum table (occuring in the first row and second column) is the sum of the corresponding entries, 4, 3, on the Eid table. (This fact has been shown in the arrangement by encircling the relevant entries).

The two brothers prepared tables of their expenditure also in the form of similar tables.

Once they had to purchase some text-books and note-books. They inquired their prices from two book-sellers in order to find out what amount would have to be spent by each and from which book-seller it would be beneficial to make the purchase. The tables prepared by them in this connection are described below:

Table of text-books and note-books.

	Books	Note-books
Rashid →	6	12
Zafar →	5	10

As usual the entries in the first row show 6 books and 12 note-books required by Rashid and the entries in the second row represent 5 text-books and 10 note-books for Zafar.

Similarly, a table showing prices was prepared like this:

	Book-seller No. 1 ↑	Book-seller No. 2 ↑
Average price per text-book →	3.50	3.75
Average price per note-book →	2.00	1.75

It can be easily seen that Rashid will get the required text-books and note-books for Rs. 45.00 from the first book-seller and for Rs. 43.50 from the second. Similarly, Zafar will get the text-books and note-books required by him from the first book-seller for Rs. 37.50 and from the second book-seller for Rs. 36.25. The following table shows these amounts:

45.00	43.50
37.50	36.25

The two brothers related this last table with the first two tables showing respectively the number of books and note-books and their prices.

$$\begin{array}{|cc|}\hline 6 & 12 \\ 5 & 10 \\ \hline \end{array} \times \begin{array}{|cc|}\hline 3.50 & 3.75 \\ 2.00 & 1.75 \\ \hline \end{array} = \begin{array}{|cc|}\hline 45.00 & 43.50 \\ 37.50 & 36.25 \\ \hline \end{array}$$

Whenever the two brothers had to prepare an account of this type, they would prepare and 'multiply' the tables in this way and they took care not to alter the order of the tables while finding the 'product'.

The entry 45 in the product table has been obtained by multiplying respectively the entries in the first row of the books table with those in the first column of the prices table and adding the product so obtained.

Answer the questions:

(i) How have the entries 43.50 and 36.25 of the product table been obtained ?

(ii) Have the entries in product table been obtained by multiplying row entries of the books table with the column entries of the prices table and adding up the products ?

(iii) What will be product table if the order of the books table and prices table are changed ?

Will the product tables in the two cases be the same or different ?

In the language of of Mathematics the above method used by Rashid and Zafar in maintaining accounts of their affairs of daily life, is called MATRIX THEORY. In mathematical terminology the tables prepared by them are of the form of MATRICES (2 x 2 matrices, to be more precise). A 2 x 2 matrix has two rows and two columns. It is read as "two by two " matrix. The four entries in the rows and columns are called ELEMENTS or COMPONENTS of the matrix.

A 2 x 2 matrix is written as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Here a, b, c, d are the elements of the matrix. The elements a, b constitute the first row and c, d make up the second row. Similarly, the elements in the first column are a, c whereas the second column comprises the elements b, d. In the present Chapter we shall deal with matrices whose elements are members of R, the set of real numbers. Therefore, in the above general form of a 2 x 2 matrix, the elements a, b, c, d denote real numbers.

Note:- Matricies are represented by capital letters of the English Alphabet, A, B, C etc.

ADDITION OF MATRICES

The method of addition of two matrices is the same as was employed by Rashid and Zafar in combining their tables. For instance, if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

$$\text{Then } A+B = \begin{bmatrix} a+p & b+q \\ c+r & d+s \end{bmatrix}$$

that is to say, for finding the sum matrix we have to add the corresponding elements of the given matrices. It is clear that

$$A + B = B + A$$

i.e. the addition of matrices possesses COMMUTATIVE PROPERTY.

By the method described above,

$$A + A = \begin{bmatrix} a + a & b + b \\ c + c & d + d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}$$

We may, therefore, take 2A to be equal to

$$2 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}.$$

In view of the above fact we generalise that for any real number k.

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

In particular, for $k = -1$,

$$(-1) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = - \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

From this, it follows that for any 2 x 2 matrix

$$A - A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The matrix on the right hand side, each of whose elements is zero, is called the ZERO-MATRIX or NULL-MATRIX. If this matrix is denoted by 0, then for every 2 x 2 matrix A, we have

$$A + 0 = 0 + A = A$$

On account of this property the null-matrix is called the IDENTITY ELEMENT w.r.t. addition of matrices. The null matrix plays the same role in the set of 2 x 2 matrices as zero is the set R of real numbers.

From the rule of addition of matrices we also conclude that two matrices will be said to be equal if and only if their corresponding elements are all equal. Thus the matrices A and B described in the beginning of the discussion will be equal if and only if

$$a = p, \quad b = q, \quad c = r, \quad d = s$$

Example:

If $A = \begin{bmatrix} 5 & -5 \\ -6 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$, find the the following matricies.

(i) A + B, (ii) −2B, (iii) B − A

Solution:

(i) $A+B = \begin{bmatrix} 5 & -5 \\ -6 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5+0 & -5+1 \\ -6+2 & 6+3 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 9 \end{bmatrix}$

(ii) $-2B = -2 \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -2\times0 & -2\times1 \\ -2\times2 & -2\times3 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -4 & -6 \end{bmatrix}$

(iii) $A-B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 5 & -5 \\ -6 & 6 \end{bmatrix} = \begin{bmatrix} 0-5 & 1-(-5) \\ 2-(-6) & 3-6 \end{bmatrix}$.

$$= \begin{bmatrix} -5 & 1+5 \\ 2+6 & -3 \end{bmatrix} = \begin{bmatrix} -5 & 6 \\ 8 & -3 \end{bmatrix}$$

EXERCISE No. 9.3

1. Add the following matrices.

(i) $\begin{bmatrix} 0 & -2 \\ -2 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

(ii) $\begin{pmatrix} 2 & 7 \\ 3 & 9 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 6 & -7 \end{pmatrix}$

2. If $A = \begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find the following matrices.

(i) $A + B$, (ii) $C + B$, (iii) $A + C$, (iv) $B + B$

(v) $-C$ (vi) $C - C$, (vii) $B - A$, (viii) $2A + 3B$,

(ix) Are $A + B = B + A$ and $C + A = A + C$?

3. Which of the following matrices are equal:

$A = \begin{pmatrix} 2 & -3 \\ 4 & 5 \end{pmatrix}, B = \begin{pmatrix} 1-1 & 0-3 \\ 4 & 10-5 \end{pmatrix}, C = \begin{bmatrix} 1+1 & 3-6 \\ 2+2 & ! + 4 \end{bmatrix}$

$D = \begin{pmatrix} 0 & -3 \\ 4 & 5 \end{pmatrix}, E = \begin{bmatrix} 4 \div 2 & 6(-\frac{1}{2}) \\ 3+1 & 5 \times 1 \end{bmatrix}$

9.5 MULTIPLICATION OF MATRICES

The rule for multiplying two matrices is also the same as employed by Rashid and Zafar in preparing product table from books and prices table.

Below we find the product AB of two matrices A, B, showing clearly the successive steps:

Step I $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap+br & \cdots \\ \cdots & \cdots \end{bmatrix}$$

Step II. $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap+br & aq+bs \\ \cdots & \cdots \end{bmatrix}$$

Step III $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & \cdots \end{bmatrix}$$

Step IV $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap+br & aq+bs \\ cp+cq & cq+ds \end{bmatrix}$$

Thus AP = $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}$$

The product AB is obtained by multiplying the elements of the rows of A with the corresponding elements of the columns of B and combining the products. Similarly, the product BA will be obtained by multiplying the elements of the rows of B with the elements of the columns of A. It is, therefore, clear that in general AB and BA will be different matrices. In other words, the product of two matrices is not commutative. As in ordinary multiplication AB means A x B and we shall stick to the first notation.

Example:-

If $A = \begin{bmatrix} 2 & -1 \\ -3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$, find BA.

Solution:-

$$BA = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} 1\times2+3(-3) & 1(-1)+3\times6 \\ 2\times2+0(-3) & 2(-1)+0\times6 \end{bmatrix}$$

$$= \begin{bmatrix} 2-9 & -1+18 \\ 4-0 & -2+0 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 17 \\ 4 & -2 \end{bmatrix}$$

EXERCISE No. 9.4

1. If $A = \begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$, find AB and BA.

2. If $C = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ $D = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, find CD and DC,

3. If $X = \begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}, Y = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

find the following matrices.

(i) XY, (ii) YX, (iii) ZY, (iv) YZ, (v) XZ, (vi) ZX.

Also find out whether

(vii) XZ = ZX, (viii) YZ = ZY = Y

4. If $A = \begin{bmatrix} 5 & -6 \\ 7 & 2 \end{bmatrix}$, prove that AO = OA where O is NULL–MATRIX.

5. Take any 2 x 2 matrix A and prove that AO and OA are zero matices.

6. If $B = \begin{bmatrix} 0 & -1 \\ 2 & 5 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, prove that

BI = IB = B

7. Take a 2 x 2 matrix A and prove that

AI = IA = A where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

From the results of questions 4, 5 and 6, 7 of the preceding exercise, we find that in the system of matrices O and I play the same

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

role as 0 and 1 play in the system of real numbers. As already stated, 0 is called null matrix. I is called UNIT MATRIX or IDENTITY MATRIX.

As already mentioned 0 is identity element with respect to addition of 2 x 2 matrices and I is the identity element with respect to multiplication.

In Question 2 of the preceding exercise two matrices are given whose product in either order is the Unit Matrix I. Such matrices are called MULTIPLICATIVE INVERSES of each other. If A and B are multiplicative inverses of each other ,B is generally written as A^{-1}. If A^{-1} is the multiplicative inverse of A, then by definition

$$AA^{-1} = A^{-1}A = 1$$

Two questions arise at this stage: (1) Does the inverse of every matrix exist? (2) If an inverse of a matrix exists, can another different inverse also exist?

At first we answer the second question. We show that any matrix can have at the most only one inverse.

If possible, let A^{-1} and A' be two inverses of a matrix A. (We shall prove that A' must be equal to A^{-1}).

Proof:

$\because$ A^{-1} is the inverse of A,

$AA^{-1} = I$.

Pre-multiplying the two sides of this equation by A'

$$A'(AA^{-1}) = A'\,1 \qquad [\because AA^{-1} = I]$$

$$= A' \quad \ldots\ldots \quad \text{(i)}$$

(I is unit matrix)

Also $\quad A'(AA^{-1}) = (A'A)A^{-1}$

$$= IA^{-1} \qquad \text{(A' is inverse of A)}$$

$$= A^{-1} \quad \ldots\ldots \quad \text{(ii)}$$

From (i) and (ii), $A' = A^{-1}$.

$\therefore$ inverse of A, if it exists, is unique.

Now we consider the second question. In order to answer it we define a special type of matrix, SINGULAR MATRIX.

A matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is said to be SINGULAR if and only if $ad - bc = 0$. Otherwise it is said to be NON-SINGULAR. The expression $ad - bc$, associated with the matrix A, will be called its DETERMINANT and will be shortly denoted as | A | (read as 'determinant of A').

As a matter of fact the inverse of a matrix exists only if it is non-singular. In other words, if a matrix, say A, is singular, its inverse does not exist.

Now consider the matrix

$$B = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

derivable from the matrix A.

We pre-multiply and post-multiply B with A.

$$AB = \frac{1}{|A|}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{|A|}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} |A| & 0 \\ 0 & |A| \end{bmatrix}$$

$$= \frac{|A|}{|A|}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\text{and } BA = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{|A|}\begin{bmatrix} da-bc & 0 \\ 0 & -cb+ad \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} |A| & 0 \\ 0 & |A| \end{bmatrix}$$

$$= \frac{|A|}{|A|}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

Thus AB = BA = 1,

i.e. $B = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is the inverse of A.

It is clear that B exists only if $|A| \neq 0$, i.e., A is non-singular.

The teacher should explain the following method of finding the inverse of the matrix.

(i) Interchange a and d

(ii) Change the signs of b and c

(iii) Divide the matrix so obtained by ad − bc, the determinant of A.

Since $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ has been proved to be the inverse of A, in future it will be denoted by A^{-1}.

Example:-

Does the multiplicative inverse of $\begin{pmatrix} 5 & 6 \\ -1 & 0 \end{pmatrix}$ exist? If it exists, find it.

Solution:-

Suppose $A = \begin{bmatrix} 5 & 6 \\ -1 & 0 \end{bmatrix}$

$$|A| = 5 \times 0 - 6(-1) = 0 + 6 = 6 \neq 0$$

Hence matrix A is non-singular and its inverse exists.

To find the inverse of A:-

(i) Interchange 5 and 0.

(ii) Change the signs of 6 and −1.

(iii) Divide the matrix obtained from (i) and (ii) by |A| or 6.

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} 0 & -6 \\ 1 & 5 \end{pmatrix} = \frac{1}{6}\begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix} = \begin{pmatrix} \frac{0}{6} & \frac{-6}{6} \\ \frac{1}{6} & \frac{5}{6} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ \frac{1}{6} & \frac{5}{6} \end{pmatrix}$$

METHOD OF VERIFICATION OF THE ANSWER

The answer will be correct if

$AA^{-1} = A^{-1}A = I$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

EXERCISE 9.5

1. Pick out singular and non-singular matrices from the following:

(i) $\begin{bmatrix} 3 & 2 \\ -2 & 0 \end{bmatrix}$, (ii) $\begin{bmatrix} 5 & 1 \\ 5 & 1 \end{bmatrix}$, (iii) $\begin{bmatrix} 2 & 8 \\ 3 & 12 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, (v) $\begin{bmatrix} -3 & 2 \\ -9 & 6 \end{bmatrix}$

2. Find the multiplicative inverses of the following matrices

(i) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$, (iii) $\begin{bmatrix} -2 & 4 \\ 3 & 8 \end{bmatrix}$

3. Pick out True and False statements from the following: (each of the matricies A and B is a 2 x 2 matrix)

(i) Multiplicative inverse of every matrix exists.

(ii) Matrix A is non-singular if $|A| \neq 0$

(iii) Matrix A is a singular matrix if $|A| = 0$

(iv) Two multiplicative inverses of a matrix A are possible.

(v) AB and BA are always equal

(vii) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is called a unit Matrix

(viii) A – A is always a zero matrix.

(ix) The addition of matrices possesses commutative property

(x) The product of two matrices is commutative.

4. If $A = \begin{bmatrix} \frac{1}{2} & 3 \\ -4 & 5 \end{bmatrix}$, find A^{-1}.

5. If $B = \begin{bmatrix} 5 & -6 \\ -1 & 2 \end{bmatrix}$, find B^{-1}.

Also prove that $BB^{-1} = B^{-1}B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

USE OF MATRICES

One use of matrices has already been mentioned. We saw in the beginning of this topic how Rashid and Zafar maintained their daily accounts in the form of matrices. Now we shall deal with an important mathematical problem in which matrices are used. This problem is that of the solution of simultaneous linear equations. We have considered a single linear equation in two variables in the beginning of this Chapter. In the system of linear equations simultaneous linear equations are quite important. By two or more simultaneous equations we mean equations which are to be true simultaneously. We have already thrown some light on the solution of a single linear equation in two variables. By drawing graph we noticed that such an equation represents a straight line. Therefore, two linear equations in two variables will represent two straight lines and the solution set of such simultaneous equations will, therefore, be the point common to the lines represented by them. In other words, the solution set of a pair of simultaneous equations will be the ordered pair corresponding to the point of intersection of the lines represented by them. For example, consider the simultaneous equations :

$$2x - y = 1,$$
$$3x + 2y = 12.$$

By the solution set of these equations we mean the point containing the ordered pair which is common to the solution sets of both the equations. One way of finding this ordered pair is to draw their graphs and to read the co-ordinates of their point of intersection. Another method is described below:

To start with, we multiply the first equation by 2* and write the other equation below it, and add the two sides of the second equation to those of the first.

$$\begin{array}{l} 4x - 2y = 2, \\ 3x + 2y = 12. \\ \hline 7x + 0 \ = 14 \end{array}$$

The last equation gives $x = 2$. We substitute this value of x in one of the given equations, say the first. Then,

$$\begin{aligned} & 2 \times 2 - y = 1, \\ \text{or} \quad & 4 - y = 1, \\ \text{or} \quad & y = 3. \end{aligned}$$

Thus the ordered pair (2, 3) is the solution of the given equations.

Next, we consider another example

$$\begin{aligned} 3x + 2y &= 5, \\ 6x + 4y &= 13. \end{aligned}$$

Multiplying the first equation by 2 and subtracting the respective sides of the second equation, we get

$$\begin{array}{l} 6x + 4y = 10 \\ 6x + 4y = 13 \\ \hline 0 \ + 0 \ = -3 \end{array}$$

---

* Here we have multiplied the first equation by 2 with a view to eliminate y by adding the respective sides of the two equations.

The last is an absurd result. Therefore, the solution set is the empty set i.e., { }

Draw the graph of these equations and see how the graph shows that the solution set of the simultaneous equations is empty.

We have considered above two examples of simultaneous equations. In the first case the solution set was non-empty and in the second it was empty. We analyse the situation by examining the matrices of the numerical co-efficients in the equations.

The matrix of co-efficients of x and y in the first set of equations is

$$A = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$$

and that of the second set is

$$B = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$$

We notice a difference between the nature of these matrices. The matrix A is non-singular whereas B is singular, i.e.,

$$|A| = 7 \neq 0,$$

$$|B| = 0$$

This is the reason why the solution set of the second set of equations is empty.

The general rule is that the solution set of a set of simultaneous equations is empty if and only if the matrix of its co-efficients is singular.

Geometrically it is true that if the co-efficient matrix of simultaneous equations is non-singular, the lines represented by them have a common point and if the matrix is singular, the lines are parallel.

By using matrices we can not only determine whether a system of equations has a non-empty or an empty solution set, but can also solve such equations . For this we have to become acquainted with 2 x 1 matrices also. As already mentioned , a 2 x 2 matrix has two rows and two columns. Such a matrix is said to be of ORDER 2 x 2. A 2 x 1 matrix is a matrix having two rows and one column. Its order is 2 x 1.

When dealing with matrices of different orders it should be remembered that :

(1) Equality relation can exist only between matrices of the same order. For example, a 2 x 2 matrics can never be equal to a 2 x 1 matrix.

(2) Addition and subtraction are possible in matrices of the same order.

(3) Product AB of matrices A, B, of different orders is, however, definable subject to the condition that the number of columns of A is equal to the number of rows of B. The reason for this condition is implied in the rule of multiplication of matrices. For finding the product AB, we have to multiply the elements of rows of A with the corresponding elements of columns of B. This is possible only if the number of entries in each row of A is the same as the number of entries in the column of B, i.e. the number of columns of A is equal to the number of rows of B.

Accordingly, if A is 2 x 2 matrix and B is any 2 x 1 matrix, then the product AB exists but the product BA does not exist.

Example

$$\text{Let } A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\text{Then } AB = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 3 + 3 \times 4 \\ 4 \times 3 + 1 \times 4 \end{bmatrix} = \begin{bmatrix} 18 \\ 16 \end{bmatrix}$$

It can be seen that

Number of columns of B = 1

Number of rows of A = 2

Since the number of columns of B ≠ number of rows of A

BA cannot be found

We illustrate the method of solving simultaneous equations by using the above principles.

For example consider the equations

$$\begin{array}{l} x \quad y = 2 \\ 3x - 3y = 5 \end{array} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$\wedge$

Whose solution has been discussed earlier also.

The matrix of the co-efficients of these equations is

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$$

The above equations can be written in the matrix form like this :

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \qquad \text{(ii)}$$

It can be easily verified that this matrix equation actually represents the above simultaneous equations. To do this we first multiply the matrices on the left hand side.

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ 3x - 3y \end{bmatrix}$$

Therefore, equation (ii) becomes

$$\begin{bmatrix} x - y \\ 3x - 3y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad \ldots \quad \text{(iii)}$$

By the definition of equality of matrices equation (iii) gives

$$x - y = 2$$
$$3x - 3y = 5$$

which are the same equations as (i).

From the above discussion we conclude that the equations:

$$\begin{aligned} ax + by &= m \\ cx + dy &= n \end{aligned} \quad \ldots \quad \text{(i)}$$

can be written in matricies in the following manner:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}$$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $L = \begin{bmatrix} m \\ n \end{bmatrix}$ ... (ii)

(ii) can be briefly written as

$$AX = L \quad \ldots \quad \text{(iii)}$$

Supposing $|A| \neq 0$, we have from equation (iii)

$$A^{-1}AX = A^{-1}L$$

i.e. $IX = A^{-1}L \qquad \because \quad AA^{-1} = I$

or $$X = A^{-1}L \qquad (\because IX = X)$$

or $$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix}$$

$$= \frac{1}{ad - bc}\begin{bmatrix} dm - bn \\ -cm + an \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{dm - bn}{ad - bc} \\ \dfrac{-cm + an}{ad - bc} \end{bmatrix}$$

From the principle of equality of matricies on both sides:

$$x = \frac{dm - bn}{ad - bc}, \quad y = \frac{-cm + an}{ad - bc}$$

i.e. the solution set of the equation (i) is

$$\left\{\left(\frac{dm - bn}{ad - bc}, \frac{an - cm}{ad - bc}\right)\right\}$$

From this discussion it also follows that if the matrix of coefficients of a system of equations is singular, its inverse does not exist and, therefore, the equation

$$AX = L$$

does not give any solution for X, i.e. the solution set of the equations is empty.

In order to clarify this method, we solve another example.

Example

Solve with the help of matrices :

$$3x - 5y = 2$$
$$x + 2y = 8$$

Solution

The matrix of the co-efficients in the equations is

$$A = \begin{pmatrix} 3 & -5 \\ 1 & 2 \end{pmatrix}$$

$$|A| = 3 \times 2 - (-5)(1) = 6 + 5$$

$$= 11 \neq 0$$

$$A^{-1} = \frac{1}{11}\begin{pmatrix} 2 & 5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{2}{11} & \frac{5}{11} \\ -\frac{1}{11} & \frac{3}{11} \end{pmatrix}$$

Therefore, given equations give

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{11} & \frac{5}{11} \\ -\frac{1}{11} & \frac{3}{11} \end{pmatrix} \begin{pmatrix} 2 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{4}{11} + \frac{40}{11} \\ \frac{2}{11} \quad \frac{24}{11} \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 2 \end{pmatrix} \quad \text{i.e.} \quad x = 4, \; y = 2.$$

∴ The solution set of the equation is $\{(4, 2)\}$

PROBLEMS BASED ON SIMULTANEOUS EQUATION

Example

Five chairs and 6 tables cost Rs.1200 and 3 chairs and 2 tables cost Rs.480. Find the cost of 1 chair and 2 tables.

[SOLUTION : Let the price of one chair be x rupees and that of a table be y rupees. According to the conditions of the equation

$$5x + 6y = 1200$$

and $$3x + 2y = 480$$

These equations can be written in the matrix form, thus

$$\begin{bmatrix} 5 & 6 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1200 \\ 480 \end{bmatrix}$$

which gives

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{8} \begin{bmatrix} 2 & -6 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 1200 \\ 480 \end{bmatrix}$$

$$= -\frac{1}{8} \begin{bmatrix} 2 \times 1200 - 6 \times 480 \\ -3 \times 1200 + 5 \times 480 \end{bmatrix}$$

$$= -\frac{1}{8} \begin{bmatrix} 2400 - 2880 \\ -3600 + 2400 \end{bmatrix}$$

$$= -\frac{1}{8} \begin{bmatrix} -480 \\ -1200 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-480}{-8} \\ \frac{-1200}{-8} \end{bmatrix}$$

$$= \begin{bmatrix} 60 \\ 150 \end{bmatrix}$$

$\therefore\ x = 60,\ y = 150$

Price of one chair = Rs.60 , Price of one table = Rs.150

Price of 1 chair and 2 tables = Rs.60 + Rs.300 = Rs.360

EXERCISE NO.9.6

1. Write down the matrices of the co-efficients of the following simultaneous equations. Find out whether the matricies are singular or non-singular.

(i) $3x + 5y = 8$
$6x + 10y = 5$

(ii) $\frac{1}{2}x - 3y = 4$
$x - 6y = 1$

(iii) $3x - 5y = 0$
$4x - 7y = \frac{1}{2}$

(iv) $3x = 2y$
$-2y = 5x + 1$

2. Show by matrices whether the following simultaneous equations possess solution sets or not.

(i) $x - y = 2$
$3x - 3y = 5$

(ii) $4x + 3y = 7$
$2x + 5y = 7$

(iii) $x = 3$
$y = 5$

(iv) $\frac{1}{2}x - y = 0$
$3x + y = 6$

3. if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

find AB and AC. Also explain with arguments whether it is possible to have BA and CA.

4. Write the following simultaneous equations in the form of matrices.

(i) $4x - 3y = 7$
$2x + 5y = 7$

(ii) $2x + 3y = 5$
$x - y = 0$

(iii) $ax + 3y = \frac{1}{2}$
$bx - dy = f$

5. Solve the following simultaneous equations by using matrices.

(i) $2x + 3y = 5$ (ii) $3x + 5y = 2$

$x - y = 0$ $2x + 4y = 3$

(iii) $5x - 2y = 7$ (iv) $x + y = 4$

$9x + 4y = 43$ $2x - 3y = 13$

6. Two footballs and three cricket balls cost Rs.140 whereas three footballs and two crikcet balls cost Rs.160. Find the price of one football and one cricket ball separately. Also find the total price of four footballs and two crikcet balls.

7. The age of a father was seven times the age of his son 7 years ago. Three years hence, the age of the father will be 3 times that of his son. Find their present ages.

8. Hamid and Nasir have certain amounts with them. If Hamid gives one rupee to Nasir, their amounts will become equal. If Nasir gives 4 rupees to Hamid his amount will become half of the amount of Hamid. Find the amount of each one of them.

9. If 1 is added to the numerator of a fraction, it becomes $\frac{2}{3}$ and if 2 is added to the denominator, it becomes $\frac{1}{2}$. Find the fraction.

10. The perimeter of a rectangle is 12 meters. If ten times the length is added to three times its breadth, the sum is 46 meters. Find the length and breadth of the rectangle.

PROPERTIES OF SIMULTANESOUS EQUATIONS AND THEIR ALGEBRAIC SOLUTION

We know that the equations in one variable

$3x + 4 = 7$

and $x = 1$

are EQUIVALENT. Two or more equations are said to be equivalent when they have the same solution set. Since the solution set of each of the above equations is (1) they are equivalent.

Similarly, the equations

$$3x + 4 + 5 = 12$$

$$3x + 2x + 4 = 7 + 2x$$

and $\quad 5x = 5$

are also equivalent to the above equations because the solution set of each one of these is { 1 }.

Actually, by solving an equation we mean to find the simplest equation equivalent to the given equation which furnishes the solution set at a glance. In the above example the simplest equation is $x = 1$ from which we at once get the solution set { 1 }.

In this connection it should be remembered that if we add a polynomial to each side of an equation the resulting equation is equivalent to the original equation. For instance, the equations

$$x - 7 = 1 - 7$$

$$3x + 2x + 4 = 7 + 2x$$

are equivalent to each of the equtions

$$x = 1$$

$$3x + 4 = 7$$

Also, by multiplying each side of an equation by a non-zero number, we get an equivalent equation. For example, the equation

$$5x = 5$$

is equivalent to the equation $x = 1$

A single linear equation in two variables satisfies the same laws, i.e., if we add a polynomial to each side of such equation or multiply each side by a non-zero number, the resulting equation is equivalent to the original equation. Because two equivalent equations have the same solution set, two equivalent equations in two variables will represent the same straight line.

The position is, however, slightly different in case of simultaneous equations. If the matrix of co-efficients of a system of two equations is non-singular, the system represents a point, i.e., the solution set of the system is an ordered pair corresponding to the point, which is the intersection of the two lines represented by the equations separately. By equivalent systems of equations we mean systems whose solution sets represent one and the same point. But the pairs of lines represented by the systems need not be the same, as it is possible for different pairs of lines to intersect in the same point. Thus the systems of equations (discussed earlier as well)

$$2x - y = 1,$$

$$3x + 2y = 12$$

and

$$x = 2$$

$$y = 3$$

are equivalent because both the pairs of lines represented by them intersect in the same point (2,3), although the lines represented by the first system are different from those represented by the second system.

Similarly, the systems of equations (also discussed earlier)

$$3x - 5y = 2,$$

$$x + 2y = 8$$

and

$$x = 4$$

$$y = 2$$

are equivalent, although the pairs of lines represented by them are quite different.

In both the above examples, the systems

$x = 2$, $y = 3$ and $x = 4$, $y = 2$

are the simplest systems equivalent respectively to the other systems.

By solving a system of equations we mean finding the simplest equivalent system.

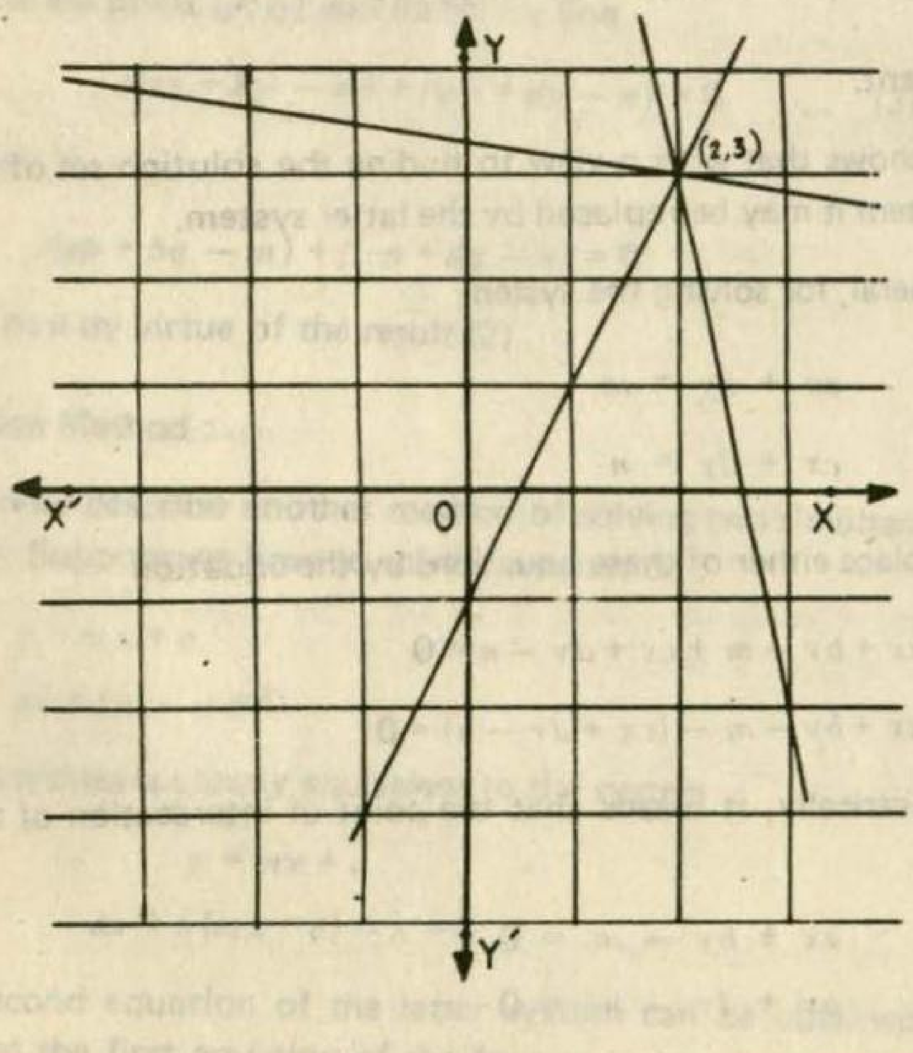
Thus the simplest system equivalent to

$$2x - y = 1$$
$$3x + 2y = 12$$

is

$$x = 2$$
$$y = 3$$

which furnishes the solution set $\{(2, 3)\}$. The adjoining graph shows the lines represented by the original system and the system $x = 2$, $y = 3$.



Both the pairs of lines intersect in the same point. On the same graph have been shown the graphs of the lines

$$2x - y = 1,$$
$$3x + 2y = 12$$

and the lines

$$2x - y - 1 + 3x + 2y - 12 = 0$$
$$2x - y - 1 - (3x + 2y - 12) = 0$$

Both the pairs of lines intersect in the same point. From this it follows that the systems

$$3x + 2y = 12, \; 2x - y = 1$$

and $2x - y - 1 + 3x + 2y - 12 = 0$

$$2x - y - 1 - (3x + 2y - 12) = 0$$

are equivalent.

This shows that with a view to finding the solution set of the former system it may be replaced by the latter system.

in general, for solving the system

$$ax + by = m$$
$$cx + dy = n$$

we may replace either of these equations by the equation

$$ax + by - m + cx + dy - n = 0$$

or by $ax + by - m - (cx + dy - n) = 0$

Geometrically, it means that the point of intersection of the lines

$$ax + by - m = 0$$
$$cx + dy - n = 0$$

lies on the line represented by the equation obtained by adding these equations or by subtracting one from the other. Not only this, the point of intersection of the above lines will lie on the line.

$$i(ax + by - m) + j(cx + dy - n) = 0$$

where i, j are any non-zero numbers. This result can be easily proved.

Proof

Let the point of intersection of the lines

$$\begin{aligned} ax + by - m &= 0 \\ cx + dy - n &= 0 \end{aligned} \qquad \cdots \qquad (1)$$

be (p, q). Then

$$\begin{aligned} ap + bq - m &= 0 \\ cp + dq - n &= 0 \end{aligned} \qquad \cdots \qquad (2)$$

Now the point (p, q) will lie on the line

$$i(ax + by - m) + j(cx + dy - n) = 0 \qquad \cdots \qquad (3)$$

if and only if

$$i(ap + bq - m) + j(cp + dq - n) = 0$$

which is true by virtue of the result (2).

Elimination Method :

We now describe another method of solving two simultaneous equations. Suppose we have to solve the equation :

$$\begin{aligned} y &= mx + c \\ ax + by - n &= 0 \end{aligned}$$

This system is clearly equivalent to the system

$$\begin{aligned} y &= mx + c \\ ax + b(mx + c) - n &= 0 \end{aligned}$$

for the second equation of the latter system can be obtained by multiplying the first equation of the former system by b and then subtracting it from the second equation of the same system.

If matrices are not to be used for solving a system of equations we can solve the system by the method described here. At page 221 we adopted this very method for solving the equation

$$2x - y = 1$$

$$3x + 2y = 12$$

More generally, suppose we have to solve the equations

$$ax + by = m$$

$$cx + dy = n$$

without using matrices. We multiply the first equation by d and the second equation by b and get

$$d(ax + by) = dm$$

$$b(cx + dy) = bn$$

Subtracting the second of these equations from the first, we have

$$x(ad - bc) = dm - bn$$

which gives $x = \dfrac{dm - bn}{ad - bc}$

(provided $ad - bc \neq 0$)

Since the last equation has been obtained in accordance with the principle proved above, it contains the point of intersection of the original equations, that is to say, the system in question is equivalent to the system

$$ax + by = m$$

$$x = \frac{dm - bn}{ad - bc}$$

If we multiply the second equation by a and subtract from the first equation, we obtain

$$by = m - a\,\frac{dm - bn}{ad - bc}$$

whence

$$y = \frac{-cm + an}{ad - bc}$$

This line should also pass through the point of intersection of the given equations. Thus the system

$$x = \frac{dm - bn}{ad - bc}$$

is equivalent to the original system. These equations being in the simplest form, give the solution set as

$$\left\{\left(\frac{dm - bn}{ad - bc}, \frac{-cm + an}{ad - bc}\right)\right\}$$

which is the same as already obtained by the matrix method

Example

Solve the following simultaneous equations :

$$2x - 3y = 5 \quad \dots\dots\dots\dots\dots\dots \quad \text{(i)}$$

$$3x + 2y = 1 \quad \dots\dots\dots\dots \quad \text{(ii)}$$

Solution

Multiplying both sides of equation (i) by 2 and equation (ii) by 3

$$2(2x - 3y) = 2 \times 5$$

$$\text{or} \quad 4x - 6y = 10 \quad \dots\dots\dots\dots\dots\dots \quad \text{(iii)}$$

$$3(3x + 2y) = 3 \times 1$$

$$9x + 6y = 3 \quad \dots\dots\dots\dots\dots\dots \quad \text{(iv)}$$

Adding both sides of equations (iii) and (iv)

$$\begin{array}{rl} 4x - 6y &= 10 \\ \underline{9x + 6y} &\underline{= \ 3} \\ 13x &= 13 \end{array}$$

$$\text{or} \quad x = 1$$

Putting this value of x in equation (i)

$$2 \times 1 - 3y = 5$$

or $2 - 3y = 5$

or $2 - 2 - 3y = 5 - 2$

or $-3y = 3$

or $y = -1$

$\therefore$ Solution set $= \{1, -1\}$

EXPLANATION

In the first step of the solution of equations, we find the value of one of the variables by eliminating the other.

(a) To eliminate y, we equalised the co-efficients of y in both the equations by multiplying equation (i) by 2 and equation (ii) by 3.

(b) $-6y$ and $+6y$ were of different signs, so we added equations (iii) and (iv) for cancellation of y. If their signs had been the same, the operation of subtraction would have become necessary for elimination of y.

(c) We found the value of y by putting the value of x in equation (i).

EXERCISE No.9.7

Find the solution sets of the following equations :

1. $x + y = 4$
   $2x - y = 5$

2. $3x - 2y = \frac{5}{2}$
   $2x - 3y = 0$

3. $9x + 4y = 43$
   $5x - 2y = 7$

4. $-3x + 4y = 0$
   $x - 4y = \frac{2}{3}$

5. $3x = 5 - 4y$
   $5y = 8 - 2x$

CHAPTER 10

GEOMETRY

FUNDAMENTAL CONCEPTS OF GEOMETRY

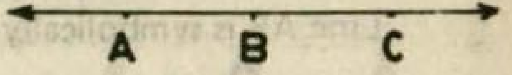
In the lower classes you have acquired some knowledge of points, lines, angles and polygons. Your previous knowledge will be, to some extent, helpful in understanding new concepts. We describe here some of the basic concepts of Geometry for revision, so that you may have convenience in learning new things.

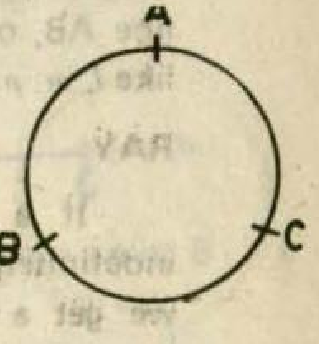
POINT

A 'POINT' is one of the fundamental concepts of Geometry. It is denoted on paper by a small dot and is usually named by a capital letter of the alphabet.

BETWEEN AND BEYOND

If three points are collinear, one of them will be BETWEEN the other two. For instance, on line AC (pictured in the adjoining figure) point B is between points A and C. If three points A, B and C are not collinear it is not possible to say that one point is between the other two. Therefore, the concept of betweenness is definite and unambiguous only for collinear points. Betweenness may be defined like this :





Definition :

If three collinear points A, B and C are such that

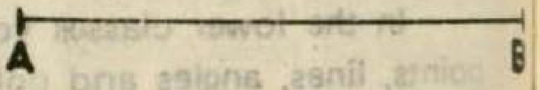
$$AC = AB + BC$$

where AC, AB, BC mean the distances between A and C, A and B, B and C respectively, then the point B is BETWEEN the points A and C.

If a point B is between points A and C, we say, point C is BEYOND B on the line AB and A is beyond B on the line BC.

LINE-SEGMENT

If A, B are two points, by the LINE-SEGMENT AB we mean the set of points consisting of the points A, B and all the points between A and B.

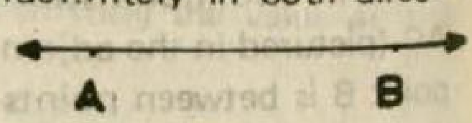


The line-segment AB is symbolically denoted as $\overline{AB}$. The points A, B are called END-POINTS of the line-segment. A line segment is named by its end-points. A line-segment is named by its end-points. The distance between the end-points A, B of a line-segment AB is the LENGTH of the segment.

The length of $\overline{AB}$ will be denoted by AB or $m\,\overline{AB}$.

STRAIGHT LINE

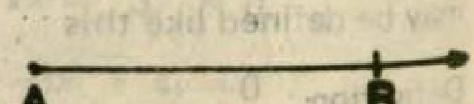
If a line-segment AB is extended indefinitely in both directions, we get a STRAIGHT LINE AB. Line AB is symbolically denoted as $\overleftrightarrow{AB}$.

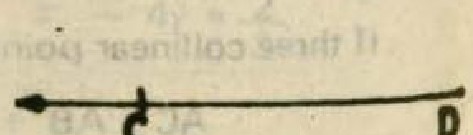


A line is named in two ways, either by any of its two points, say line AB, or by a single (small) letter like *l*, *m*, *p* etc.

RAY

If a line-segment is extended indefinitely only in one direction, we get a RAY. For instance, if a line-segment AB is extended indefinitely beyond B, we get the ray AB. Similarly, if we extend a line-segment CD beyond C, we get another ray DC.





Point A is the END-POINT of ray AB and point D is the end-point of ray DC

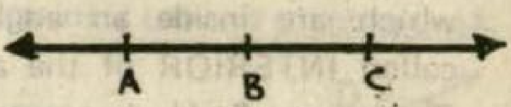
Symbolically, ray, AB is denoted as $\overrightarrow{AB}$ and ray DCas $\overrightarrow{DC}$.

It should be remembered that a line-segment has two end-points, a ray has only one end point, but a line has no end-point.

A ray is named by two of its points, one of them being its end-point. The end-point is named first.

OPPOSITE RAYS

If A, B, C are three collinear points (points lying on the same line) such that B is between A and C, then $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are said to be OPPOSITE RAYS. In other words, two rays are opposite rays if and only if :

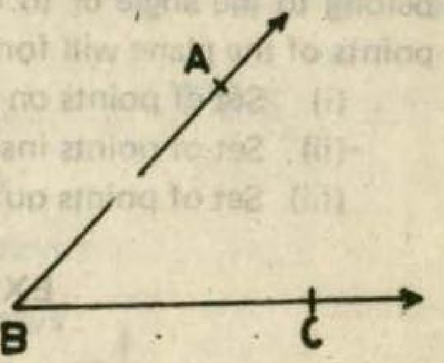


(i) they are collinear, (ii) they have a common end-point, and (iii) no other point is common to them.

ANGLE

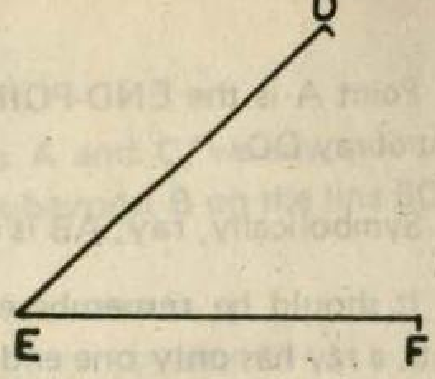
In the adjoining figure are shown two rays, BC and BA. They have a common end-point B, and they are not collinear. The union of these rays is an ANGLE called angle ABC, or angle CBA or simply angle B.

Symbolically, angle ABC is denoted as ∠ ABC.



The rays BA, BC are the ARMS of this angle. The point B is the VERTEX of the angle.

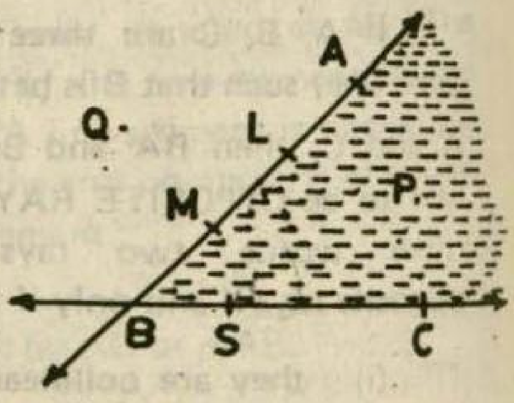
$\overline{EF}$ and $\overline{ED}$ are coterminous (having the same end-point) and non-collinear line-segments. By extending EF indefinitely beyond F and ED beyond D, we



can get coterminous and non-colinear rays EF, ED. Therefore, we may say that the line-segments EF, ED DETERMINE an angle and the angle determined by them is also named as angle DEF.

INTERIOR AND EXTERIOR OF AN ANGLE

In the adjoining figure points A, P are on the same side of line BC and points C, P are on the same side of line AB. In such a case we say that point P is inside angle ABC.



The set of all those points which are inside an angle, is called INTERIOR of the angle. The points S, M, L are on the angle but do not belong to the interior of the angle. The point Q is neither inside the angle nor on the angle,but it is OUTSIDE the angle. The set of all those points which are outside an angle is called the EXTERIOR of the angle.

N.B. Given an angle, any point of the plane will either belong to the angle or to its interior or to its exterior. That is,the points of the plane will form the following three disjoint sets

(i) Set of points on the angle ;
(ii) Set of points inside the angle ;
(iii) Set of points outside the angle.

EXERCISE NO.10.1

1. In figure (1) which set of points is a (a) line, (b) line-segment, and (c) ray ?

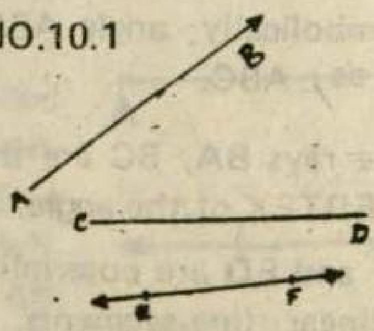
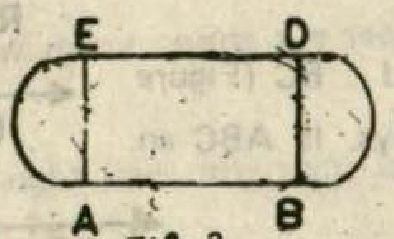


Fig. 1

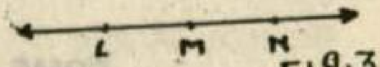
2. Draw figures to present $\overrightarrow{QP}$, $\overleftrightarrow{ST}$, $\overrightarrow{PQ}$, $\overline{LM}$.

3. Name the line-segments contained in figure (2)



Fig·2

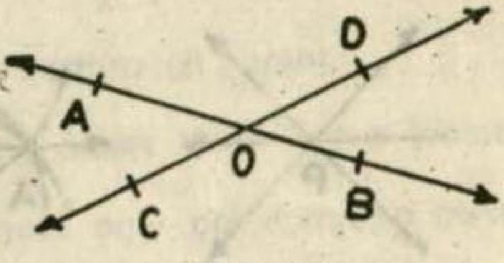
4. In figure (3) some collinear rays have been shown. Name all the rays having L, M or N as their end points.



Fig·3

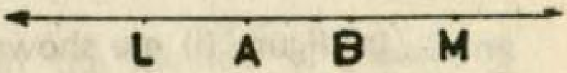
5. If $\overrightarrow{AB}$, $\overrightarrow{AC}$ be opposite rays, which of the points A, B, C is between the other two, and which point is beyond which point?

6. Lines AB , CD intersect at a point O, (Figure 4). Name the two pairs of opposite rays so formed.



Fig·4

7. Figure (5) contains some collinear rays. Give reasons in support of the following statements :



Fig·5

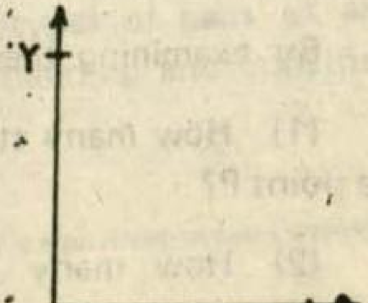
(i) $\overrightarrow{AM}$ and $\overrightarrow{AL}$ are opposite rays ;

(ii) $\overrightarrow{BM}$ and $\overrightarrow{AL}$ are not opposite rays ;

(iii) $\overrightarrow{AM}$ and $\overrightarrow{BL}$ are not opposite rays ;

(iv) $\overrightarrow{LA}$ and $\overrightarrow{LM}$ are the same ray

8. In Figure (6) $\overrightarrow{PX}$ and $\overrightarrow{PY}$ are coterminous rays but they are not opposite rays. Why ?



Fig·6

9. Are $\overrightarrow{RA}$ and $\overrightarrow{BQ}$ (Figure 7) opposite rays ? Give reasons for your answer.

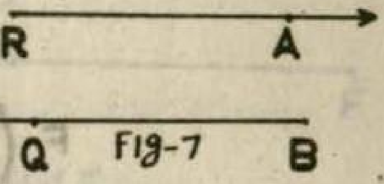


Fig-7

10. $\overrightarrow{BA}$ and $\overrightarrow{BC}$ (Figure 8) are opposite rays. Is ABC an angle ? If not, why ?

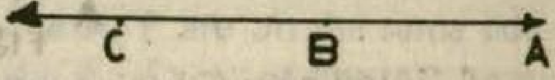
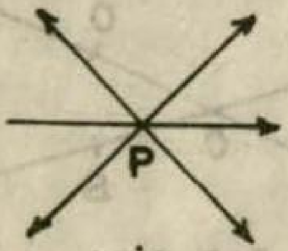
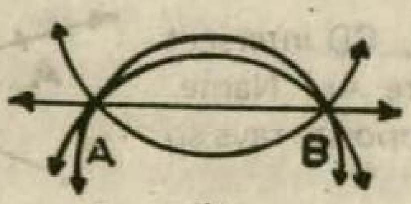


Fig. 8

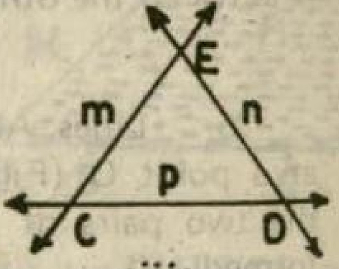
SOME PROPERTIES OF POINTS AND LINES



(i)



(ii)



(iii)

In figure (i) are shown some straight lines passing through a point P.

In figure (ii) are shown some lines passing through two points A and B.

In figure (iii) are shown three straight lines, *m,n,p* which Intersect two by two.

By examining these figures answer the following questions:

(1) How many straight lines can be made to pass through the point P?

(2) How many lines of figure (ii) are straight lines?

(3) Can a straight line pass through two points ?

(4) How many straight lines can, at the most, pass through A and B ?

(5) Atleast how many points are required to determine a straight line ?

(6) If two straight lines intersect, how many points will be common to them ?

N. B. Henceforth, 'line' will mean 'straight line' unless otherwise specified.

From the above discussion, we get the following

Important Results :

1. Infinite number of lines pass through a point.
2. One and only one* line passes through two points.
3. Two intersecting lines have only one common point.

KINDS OF ANGLES

You are already familiar with the method of measuring an angle with the help of a protractor, and you know that the degree measure of an angle is always between 0 and 180 degrees. An angle of less than 90 degrees is called an ACUTE ANGLE, an of 90 degrees is a RIGHT ANGLE and an angle of more than 90 degrees is an OBTUSE ANGLE.

We now describe certain special types of pairs of angles. These types will be useful in understanding and constructing a number of geometrical fingures.

---

* The statement 'one and only one line passes' is equivalent to two statements:

(i) One line passes;

(ii) Not more than one line pass.

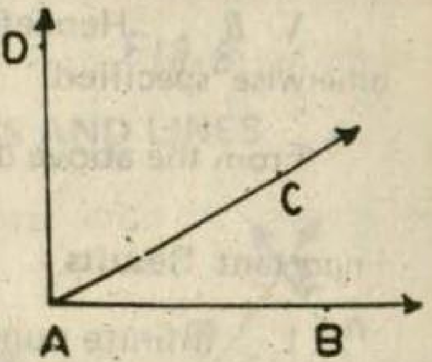
Complementary Angles :

If the sum of the measures of two angles is 90, they are called complementary angles. Each of the angles is called COMPLEMENT of the other.

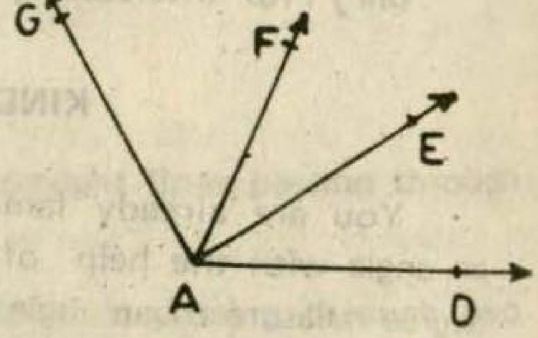
Supplementary Angles

If the sum of the measures of two angles is 180, they are called supplementary angles. Each of the angles is called SUPPLEMENT of the other.

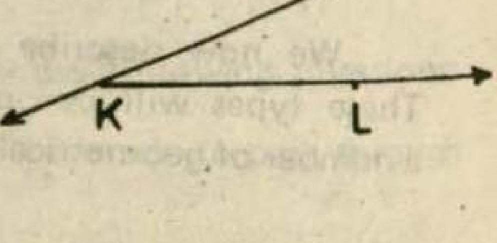
Adjacent Angles



If two angles are such that (i) they have a common vertex, (ii) they have a common arm, and (iii) their interiors are disjoint, they are called adjacent angles. In the adjoining figure ∠ CAB and ∠ DAC are adjacent angles, but ∠CAB and ∠DAB are not adjacent angles (why ?)

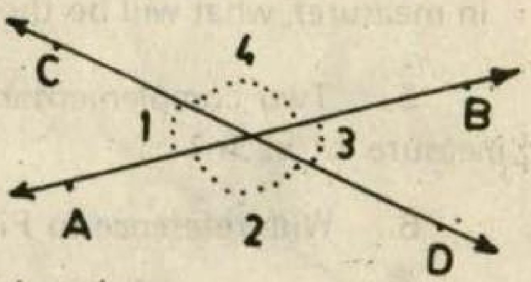


∠ EAD and ∠FAE are adjacent angles, ∠ EAD and ∠GAF are not adjacent angles.



Which arm of ∠AKL and ∠MAK is common ? Have their interiors any common point ? Are these angles adjacent angles ? If not, why ?

Vertical Angles:



Two angles whose arms are two pairs of opposite rays are called vertical angles. In the adjoining figure ∠ 1 and ∠ 3 are one pair of vertical angles. Similarly ∠2 and ∠4 are another pair of vertical angles.

EXERCISE No. 10.2

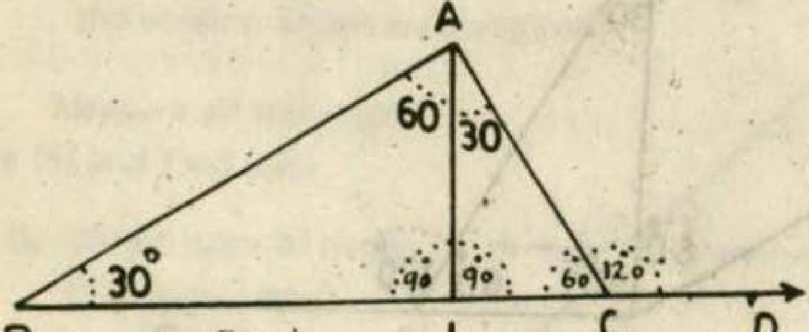


Fig. I

1. Examine the above figure (fig. 1) and

   (i) Name four pairs of complementary angles ;

   (ii) Name three pairs of supplementary angles ;

   (iii) Name the complement of ∠ACL ;

   (iv) Name the suplement of ∠ALC ;

   (v) Name the adjacent supplementary angles ;

   (vi) Name a pair of angles which are supplementary but not adjacent;

   (vii) In Q. (v) what did you discover about the outer arms of the adjacent supplementary angles ?

2. Is each of the complementary angles an acute angle ?

3. Can both of the two supplementary angles be (i) acute, (ii) obtuse ?

4. If two supplementary angles are congruent (equal in measure), what will be the measure of each ?

5. Two complementary angles are congruent, what is the measure of each ?

6. With reference to Figure (2),

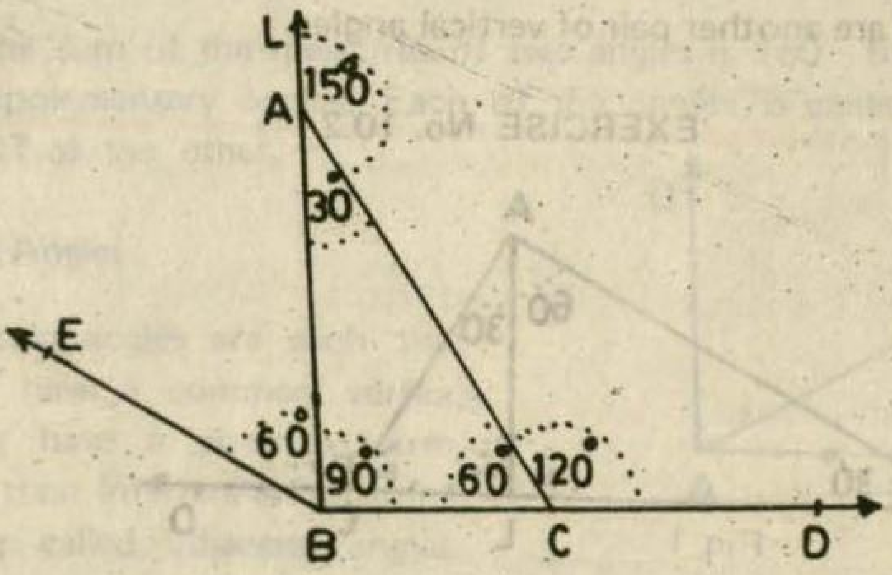


Fig. 2

(i) Name two pairs of angles which are adjacent and supplementary. Are their outer arms collinear ?

(ii) Name a pair of angles which are adjacent but not supplementary. Are the outer arms of these angles collinear ?

(iii) What conclusion can you draw from the answer to Q. (i) and (ii) ?

7. In figure (3).

(i) Measure $\angle 1$, $\angle 2$ and find $m\angle 1 + m\angle 2$. Are these supplementary angles ?

(ii) Measure $\angle 3$, $\angle 4$. Are these supplementary angles?

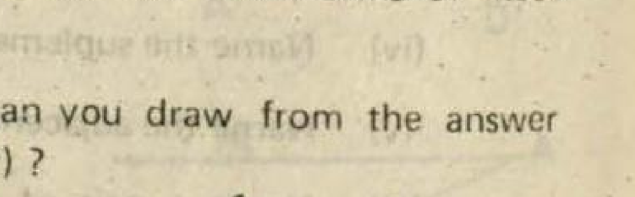


Fig. 3

8. $\overleftrightarrow{AB}$ and $\overrightarrow{CD}$ intersect at a point P (Figure 4),

(i) Name two pairs of vertical angles;

(ii) Name two pairs of adjacent angles:

(iii) Measure the remaining angles and verify that the vertical angles are congruent.

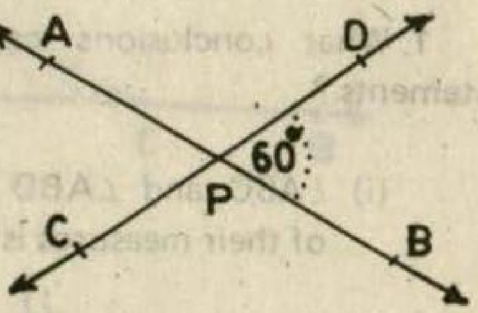


Fig. 4

9. Measure all the angles in figure (5) and find out:

(i) Which pairs of angles are congruent ?

(ii) Which pairs of angles are supplementary ?

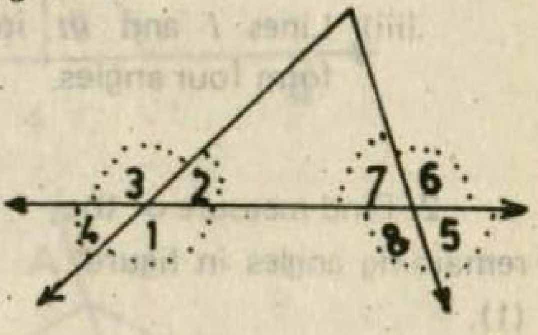


Fig. 5

(iii) What conclusion can you draw about vertical angles ?

IMPORTANT RESULTS ABOUT ANGLES

1. If two adjacent angles are supplementary, their outer arms are collinear.

2. If the outer arms of two adjacent angles are collinear (form a pair of opposite rays), they are supplementary.

3. A pair of vertical angles are congruent.

N. B. : If two angles are adjacent as well as supplementary they are called ADJACENT SUPPLEMENTARY ANGLES.

EXERCISE No. 10.3

1. What conclusions can you draw from the following statements ?

(i) ∠ABC and ∠ABD are adjacent angles and the sum of their measures is 180°.

(ii) Point B belongs to a line AC and lies between A and C, and BD is a ray which does not lie on $\overrightarrow{AB}$.

(iii) Lines l and m intersect at a point O and thus form four angles.

2. Find measure of the remaining angles in figure (1).

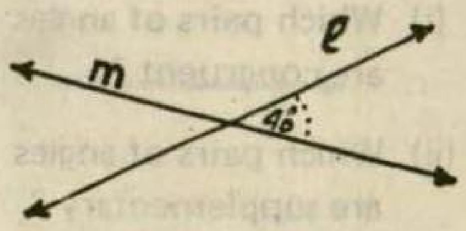


Fig. 1

3. In figure (2) angle 1 and 2 are congruent. Are angles 3 and 4 also congruent ?

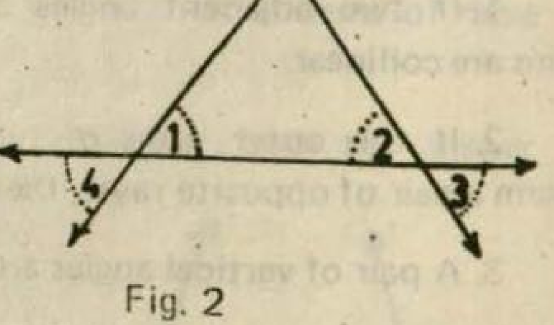


Fig. 2

4. In figure (3), $m\angle ACD = 24°$. Find $m\angle DCB$.

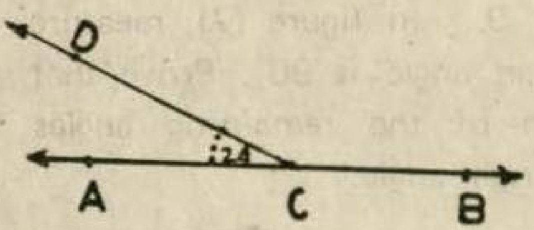


Fig. 3

5. In figure (4), $m\angle 1 = m\angle 2$. Prove that $\overrightarrow{ML} \perp \overleftrightarrow{PQ}$

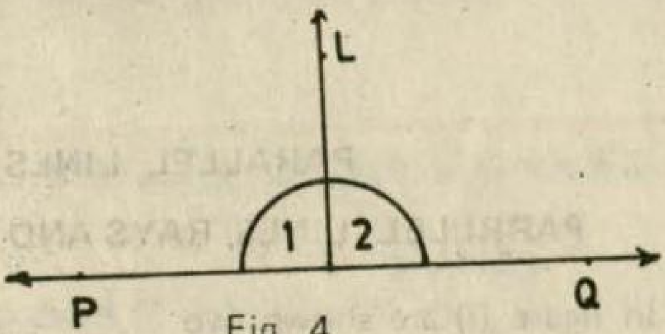


Fig. 4

6. In figure (5), measure of $\angle ACB$ is 70° and measure of $\angle ACD$ is 110°. Prove that $\overrightarrow{CB}$ and $\overrightarrow{CD}$ are collinear rays.

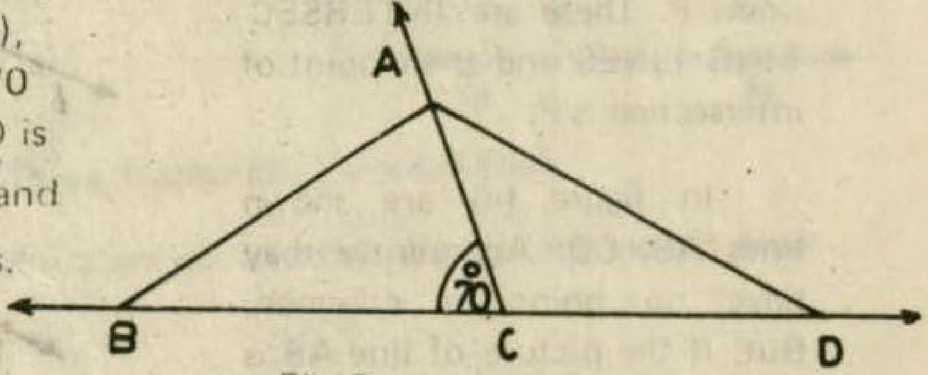


Fig. 5

7. The two arms of an iron-bar are inclined to each other at an angle of 45°. Through what angle should one of the arms be rotated to make the two arms collinear?

8. In figure (6) $\angle ABC \cong \angle ACB$, prove that the measures of $\angle x$ and $\angle y$ are equal.

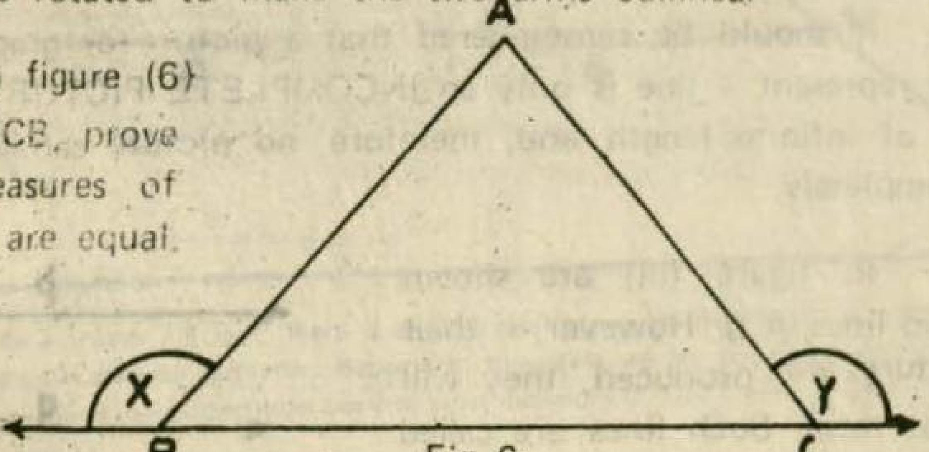


Fig. 6

9. In figure (7), measure of an angle is 90°. Prove that each of the remaining angles is a right angle.

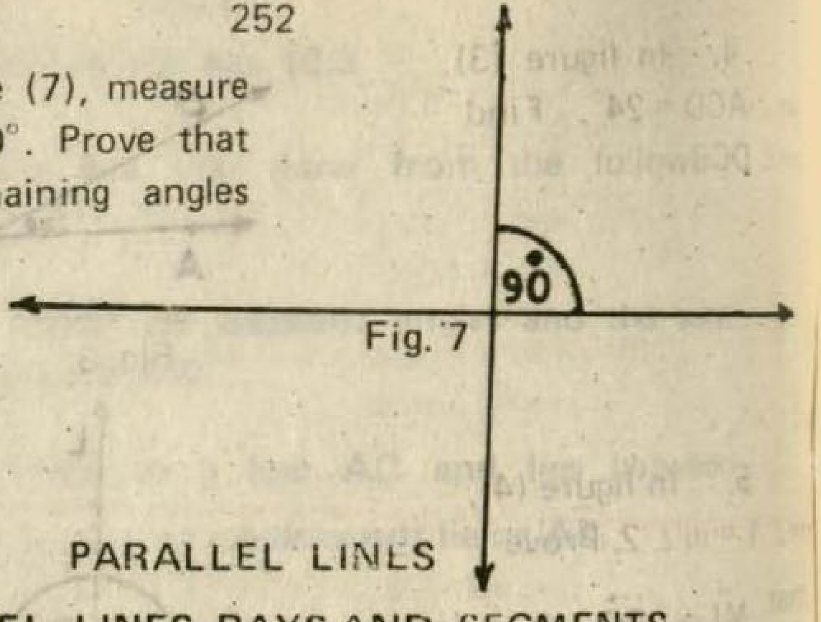
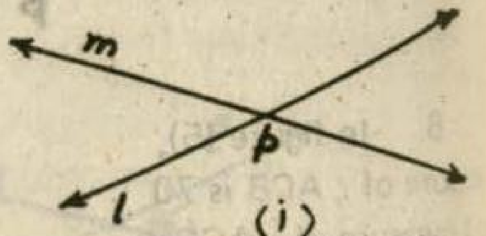


Fig. 7

PARALLEL LINES

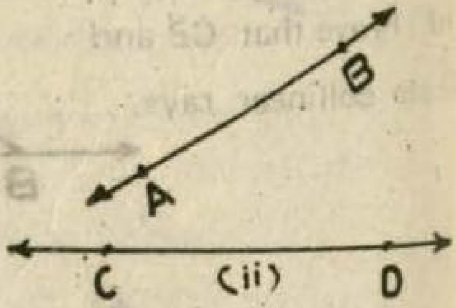
PARRLLEL LINES, RAYS AND SEGMENTS

In figure (i) are shown two lines, *l*, *m*. They have a common point P. These are INTERSECTING LINES and their point of intersection is P.



(i)

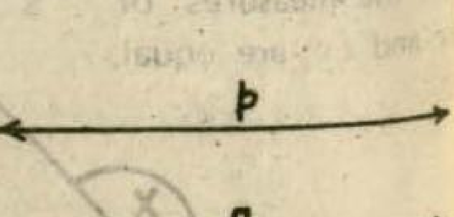
In figure (ii) are shown lines AB, CD. Apparently thay have no point in common. But, if the picture of line AB is sufficiently produced beyond A, it will intersect CD. Therefore in this case also the lines AB, CD are regarded as intersecting lines.



(ii)

It should be remembered that a picture or graph drawn to represent a line is only an INCOMPLETE PICTURE. A line is of infinite length and, therefore, no picture can represent it completely.

In figure (iii) are shown two lines, *p*, *q*. However, if their pictures are produced, they will not meet. Such lines are called PARALLEL LINES. In other words,



(iii)

Parallel lines are such lines as do not intersect*.

We may also say that parallel lines are those lines whose intersection set is an empty set.

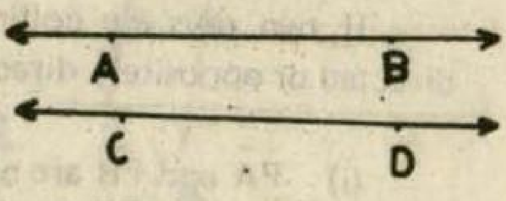
Parallel Rays

If two rays lie on parallel lines, they are said to be parallel.

Parallel Line-Segments

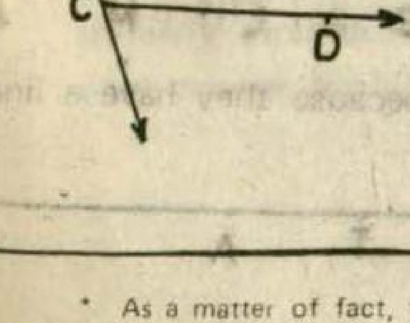
If two line-segments lie on parallel lines, they are also called parallel segments.

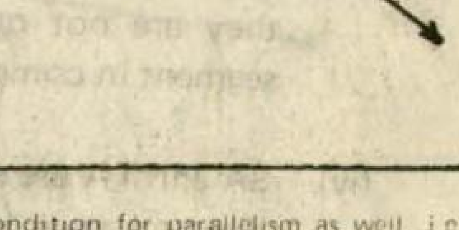
If AB, CD are parallel lines we write $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and read it as 'line AB is parallel to line CD.'



Similarly Directed Rays, Oppositely Directed Rays

In case of parallelism of two rays, two situations can arise:





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[*] As a matter of fact, there is another condition for parallelism as well, i.e., the lines should also be coplanar. Because in this class we are concerned with plane figures only, the second condition has not been mentioned. The teacher should explain this.

(i) Both the rays are on the same side of the line passing through their end-points, for instance, $\vec{AB}$, $\vec{CD}$.

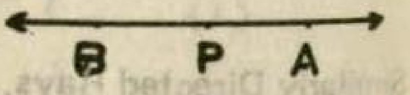
(ii) The rays lie on opposite sides of the line joining their end-points, for example, $\vec{EF}$, $\vec{HG}$.

$\vec{AB}$ amd $\vec{CD}$ are oppositely directed rays. They are parallel and lie on the same side of the line passing through their end-points.

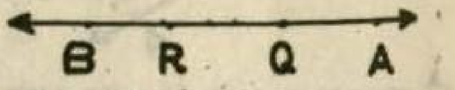
$\vec{EF}$ and $\vec{HG}$ are oppositely directed rays. They are parallel and lie on opposite sides of the line passing through their end-points.

If two rays are collinear, even then they may be similarly directed or oppositely directed rays.

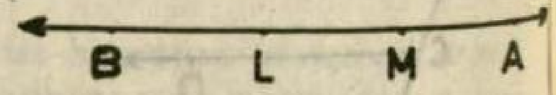
(i) $\vec{PA}$ and $\vec{PB}$ are opposite rays. They are also oppositely directed rays.



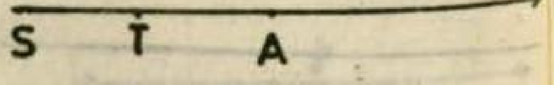
(ii) $\vec{QA}$ and $\vec{RB}$ are oppositely directed rays. But they are not opposite rays because their end-points are not common.



(iii) $\vec{LA}$ and $\vec{MB}$ are oppositely directed rays but they are not opposite rays because they have a line-segment in common.



(iv) $\vec{SA}$ and $\vec{TA}$ are similarly directed rays. Here TA is contained in (is a sub-set of) $\vec{SA}$.



EXERCISE No. 10.4

1. What is the criterion for two lines to be parallel ?

2. What is the criterion for two line-segments to be parallel ?

3. Are opposite rays always oppositely directed ?

4. Look at the adjoining figure and tell which of the pairs of rays are similarly directed, oppositely directed or opposite rays :

(i) $\overrightarrow{AB}$, $\overrightarrow{CD}$;

(ii) $\overrightarrow{FE}$, $\overrightarrow{GH}$;

(iii) $\overrightarrow{LP}$, $\overrightarrow{MO}$;

(iv) $\overrightarrow{MP}$, $\overrightarrow{LP}$;

(v) $\overrightarrow{MO}$, $\overrightarrow{LO}$;

(vi) $\overrightarrow{MO}$, $\overrightarrow{MP}$.

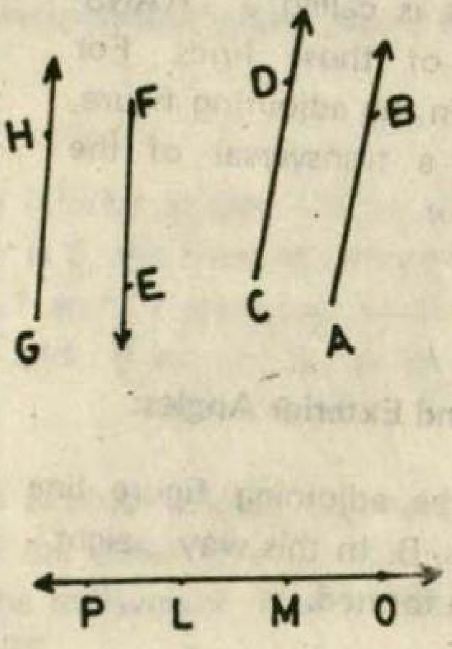


Fig 1

5. Tell whether $\overrightarrow{XY}$ and $\overrightarrow{ST}$ in figure (2) are similarly directed. Justify your answer with reasons.

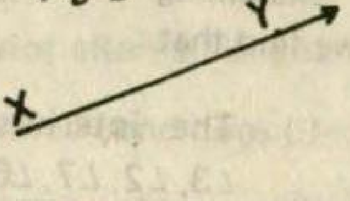


Fig. 2

6. $\overrightarrow{LM}$ and $\overrightarrow{ML}$ (figure 3) are not opposite rays. Why? Are these oppositely directed rays?

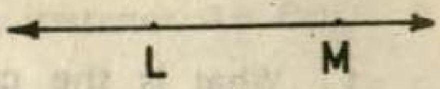
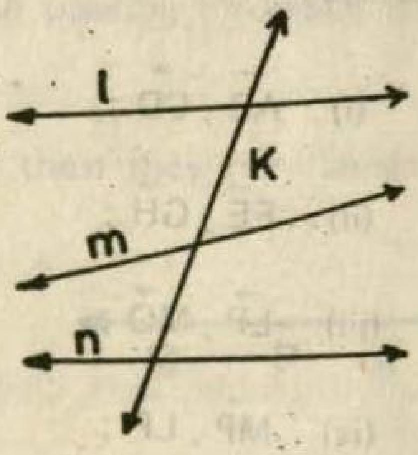


Fig. 3

2.2 ALTERNATE ANGLES, CORRESPONDING ANGLES

Transversal

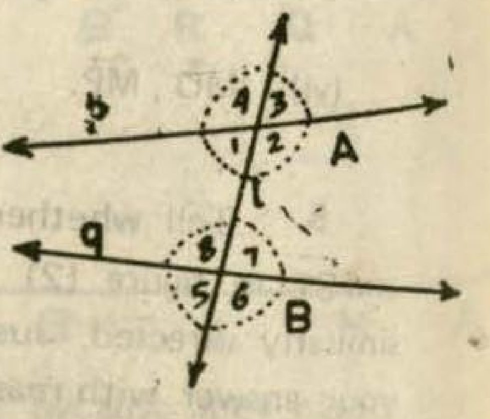
When one line intersects two or more other lines, at different points, it is called a TRANSVERSAL of those lines. For example, in the adjoining figure, line k is a transversal of the lines l, m, n.



Interior and Exterior Angles:

In the adjoining figure line t intersects the lines p, q at points, A, B. In this way eight angles are formed.

On examining the figure closely, we find that :



(i) The interiors of ∠3, ∠2, ∠7, ∠6 are on one side of the transversal and the interiors of ∠4, ∠1, ∠8, ∠5 are on its other side.

(ii) Both the points of intersection A, B lie on the arms of ∠2. This is an INTERIOR ANGLE. Similarly, ∠1, ∠7, ∠8 are interior angles.

(iii) ∠3 contains only one point of intersection, A. This is an EXTERIOR ANGLE of the lines p, q. Similarly ∠4, ∠5 and ∠6 are exterior angles.

Corresponding Angles:

As already seen, four of the angles in the above figure are interior angles and the remaining four are exterior angles.

∠7 is an interior angles and ∠3 is an exterior angle. Interiors of both these angles are on the same side of the transversal. This is a pair of CORRESPONDING ANGLES. In the figure there are three other pairs of corresponding angles. Name them.

Alternate Angles :

∠1 and ∠7 are both interior angles. Vertex of one of them is A and that of the other is B, and their interiors are on opposite sides of the transversal. ∠1 and ∠7 are a pair of INTERIOR ALTERNATE ANGLES. ∠2 and ∠8 are also a pair of interior alternate angles.

∠3 and ∠5 are both exterior angles. The vertex of one of them is at A and that of the other is at B. Also their interiors are on opposite sides of the transversal. This is a pair of EXTERIOR ALTERNATE ANGLES.

Name the other pair of exterior alternate angles.

Alternate angles must fulfil three conditions:

(i) Either both the angles are interior angles or both are exterior angles.

(ii) They do not have a common vertex.

(iii) The interiors of the angles are on opposite sides of the transversal.

Henceforth, by 'alternate angles' we shall mean interior alternate angles.

N. B. : The letter Z or its reverse Ƨ is helpful in recognising alternate angles.

EXERCISE No. 10.5

1. In how many points will a transversal meet two lines?

2. A line *l* intersects two lines *p*, *q* in points A, B (figure 1). Look at the figure and name the following pairs of angles.

(i) Interior alternate angles.

(ii) Exterior alternate angles.

(iii) Corresponding angles.

(iv) A pair of adjacent supplementary angles.

(v) A pair of vertical angles.

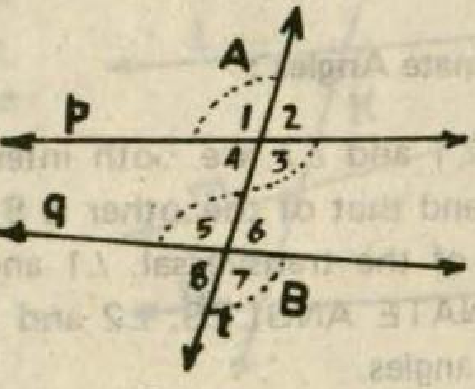


Fig · 1

3. Look at figure (2) and answer the following questions:

(i) The line *p* is a transversal of which lines?

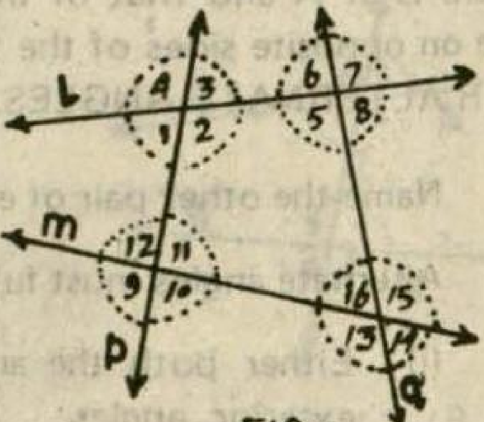


Fig.2

(ii) Name four interior and four exterior angles formed by the lines *p*, *l* and *m*.

(iii) Of which lines is *m* a Transversal?

(iv) Name one pair of corresponding angles, one pair of interior alternate angles and one pair of exterior alternate angles formed by the lines m and p, q.

4. With reference to figure (3) state the kinds of the following pairs of angles:

(i) $\angle 1, \angle 3$;
(ii) $\angle 3, \angle 4$;
(iii) $\angle 3, \angle 7$;
(iv) $\angle 4, \angle 6$;
(v) $\angle 1, \angle 7$;

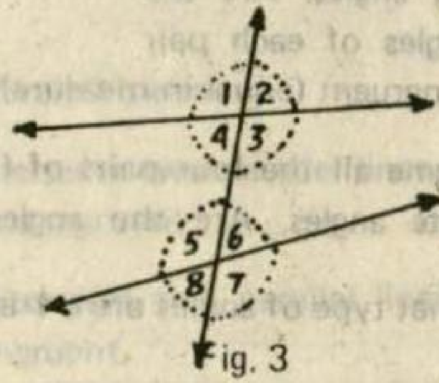


Fig. 3

2.3 PLAYFAIR'S AXIOM AND PARALLEL LINES

In every deductive system some statements are accepted as true, without proof, in the beginning. Such statements are called AXIOMS or POSTULATES. One of the important postulates of geometry is PLAYFAIR'S AXIOM. Though a point, not on a given line, there passes ONE AND ONLY ONE line which is parallel to the given line.

This postulate is ascribed to the famous Scottish mathematician John Playfair (1748–1819).

'There passes one and only one line' means that (i) at least one line passes, and (ii) not more than one lines pass.

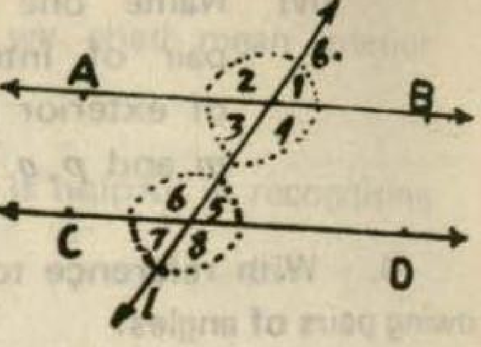
This postulate tells us that

'Two intersecting lines cannot be parallel to one and the same line.'

Corresponding Angles and Alternate Angles of Parallel Lines.

Lines AB, CD are parallel and a line l intersects them and $\angle 1$ is of 60°.

(i) Measure the remaining angles with a protractor.



(ii) Name all the four pairs of corresponding angles. Are the angles of each pair congruent (equal in measure) ?

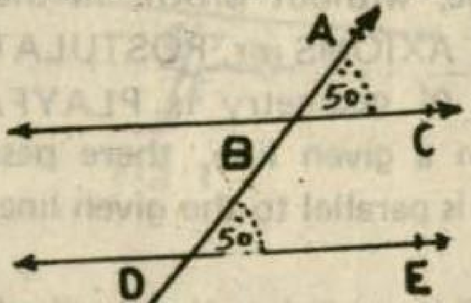
(iii) Name all the four pairs of (interior and exterior) alternate angles. Are the angles of each pair congruent?

(iv) What type of angles are $\angle 4$ and $\angle 5$?

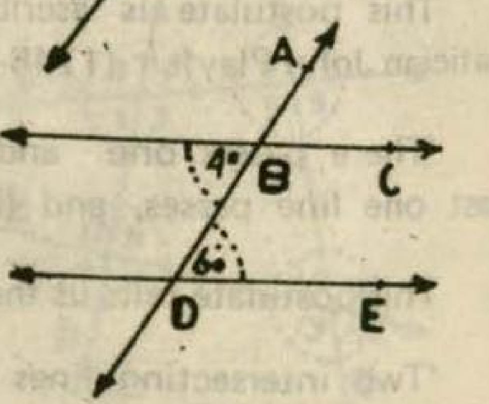
Is $m\angle 4 + m\angle 5 = 180^\circ$? What is the sum of the measures of the remaining two interior angles on the same side of the transversal ?

What conclusions can you draw from the above exercise ?

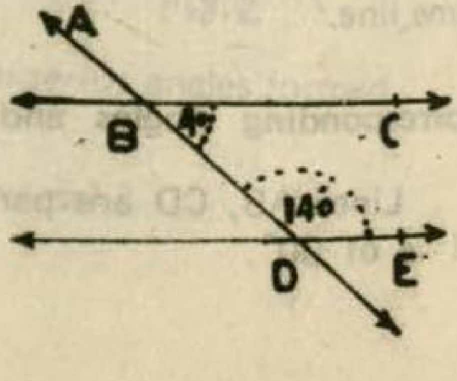
On a sheet of paper draw two corresponding angles $\angle ABC$, $\angle ADE$ of equal measure. Produce $\overline{BC}$ and $\overline{DE}$ in both directions. Do $\overleftrightarrow{BC}$, $\overleftrightarrow{DE}$ appear to meet ?



Next repeat the experiment by drawing congruent alternate angles.



Finally repeat the experiment by drawing two lines and a transversal such that the sum of the measures of a pair of interior angles on the same side of the transversal is 180°



You should have noticed that in each case the two lines are parallel.

Therefore, in the light of the above experiments, we get the following results :

Important Theorems about Parallel Lines :

1. If a transversal intersects two parallel lines, the corresponding angles so formed are congruent.

2. If a transversal intersects two parallel lines, the alternate angles so formed are congruent.

3. If a transversal intersects two parallel lines, a pair of interior angles on the same side of the transversal are supplementary.

4. If a transversal intersects two lines such that a pair of corresponding angles are congruent, the lines are parallel.

5. If a transversal intersects two lines such that a pair of alternate angles are congruent, the lines are parallel.

6. If a transversal intersects two lines such that a pair of interior angles on the same side of the transversal are supplementary, the lines are parallel.

EXERCISE No. 10.6

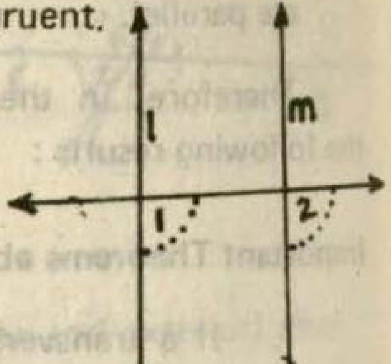
1. What conclusion can you draw from each of the following statements :

(i) A line AB is parallel to a line CD and a line l intersects $\overleftrightarrow{AB}$.

(ii) A line AB is parallel to a line CD and the line CD is a parallel to line EF.

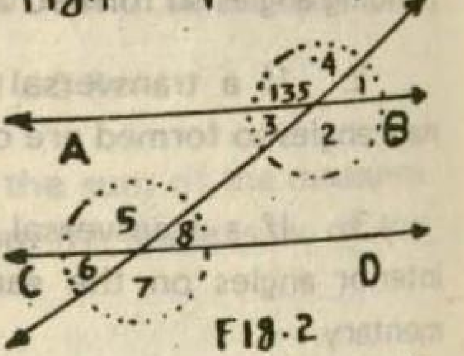
(iii) A line AB is parallel to a line CD and $\overleftrightarrow{EF}$ intersects them.

(iv) A line l intersects lines AB, CD such that a pair of alternate angles are congruent.



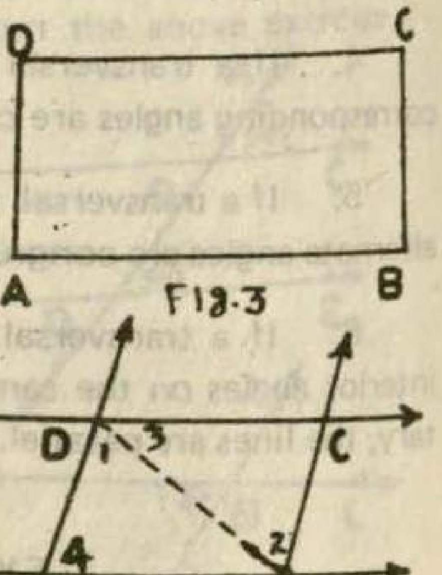
Fig·1

2. Lines l, m are parallel. What can you say about $\angle 1$ and $\angle 2$? (Figure 1).



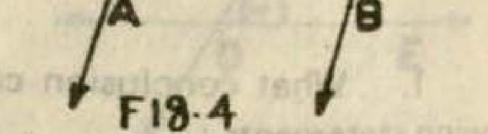
Fig·2

3. Lines AB, CD are parallel (figure 2). One of the angles, as shown, is of 135°. Find the measures of the remaining angles.



Fig·3

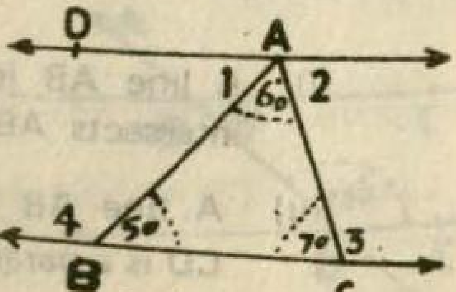
4. In a parallelogram ABCD one angle is a right angle. Prove that its remaining angles are also right angles.



Fig·4

5. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. Prove that

$m\angle 1 = m\angle 2$, $m\angle 3 = m\angle 4$.



Fig·5

6. In figure (5) $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. Find the measures of $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$.

7. The arms of angles ABC and PQR two pairs of similarly directed rays. Show that the angles are congruent.

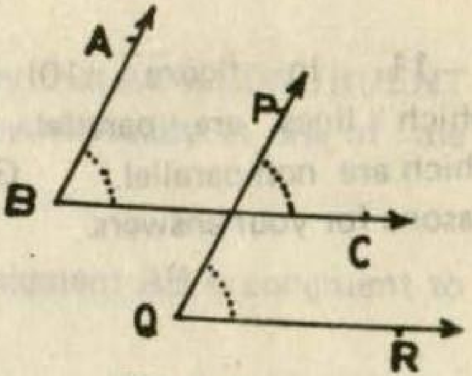


Fig. 6

8. One pair of the arms of angles LMN and XYZ are similarly directed rays and the other pair are oppositly directed rays. Show that the angles are supplementary.

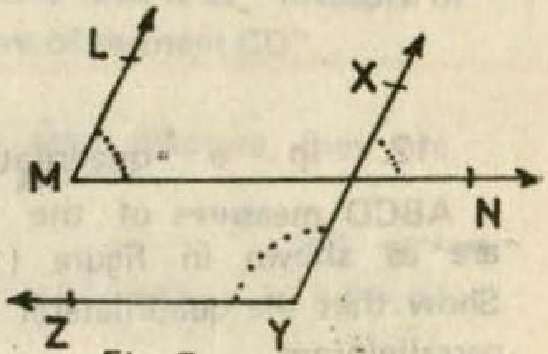


Fig. 7

9. In figure (8) prove that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AC} \parallel \overleftrightarrow{BD}$.

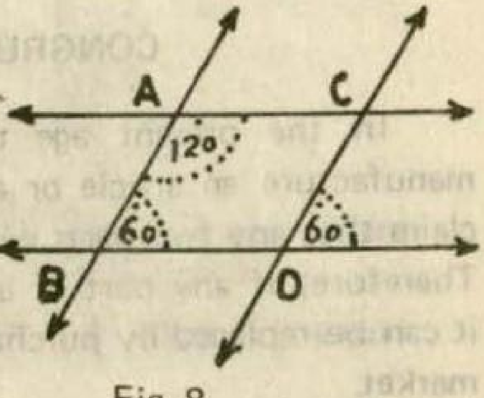


Fig. 8

10. LM is perpendicular to both the lines AB and CD. Show that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

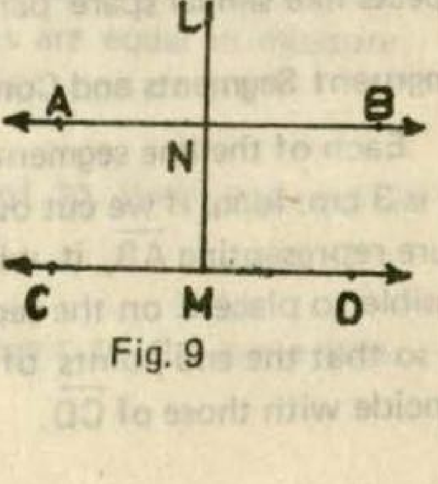


Fig. 9

11. In figure (10) tell which lines are parallel and which are non-parallel. Give reasons for your answers.

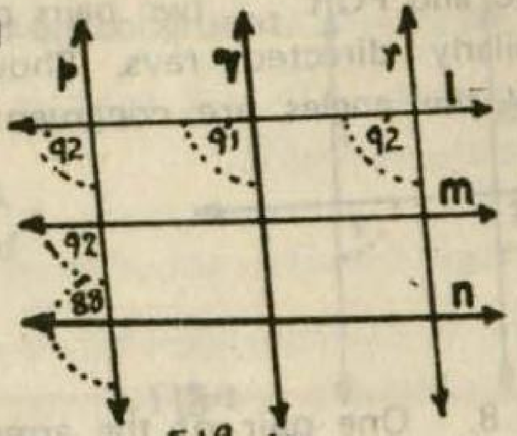


Fig. 10

12. In a quadrilateral ABCD measures of the angles are as shown in figure (11). Show that the quadrilateral is a parralleloram.

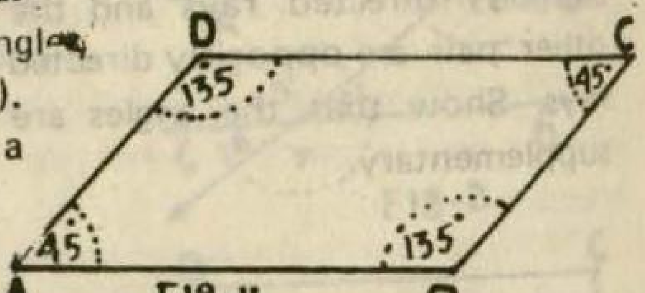


Fig. 11

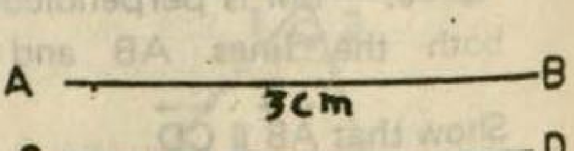
CONGRUENT TRAINGLES

In the present age of technology industrialists usually manufacture an article or a part of a machine in lacs and they claim that any two parts will be exactly similar in size and shape. Therefore, if any part of a machine or wagon becomes useless, it can be replaced by purchasing an exactly similar part from the market.

In this Section we shall study about geometrical figures which are equal in size and shape, i.e., which are equal in all respects like similar spare parts of a machine.

Congruent Segments and Congruent Angles

Each of the line segments AB, CD is 3 cm. long. If we cut out the figure representing $\overline{AB}$, it will be possible to place it on the segment CD so that the end-points of $\overline{AB}$ coincide with those of $\overline{CD}$.



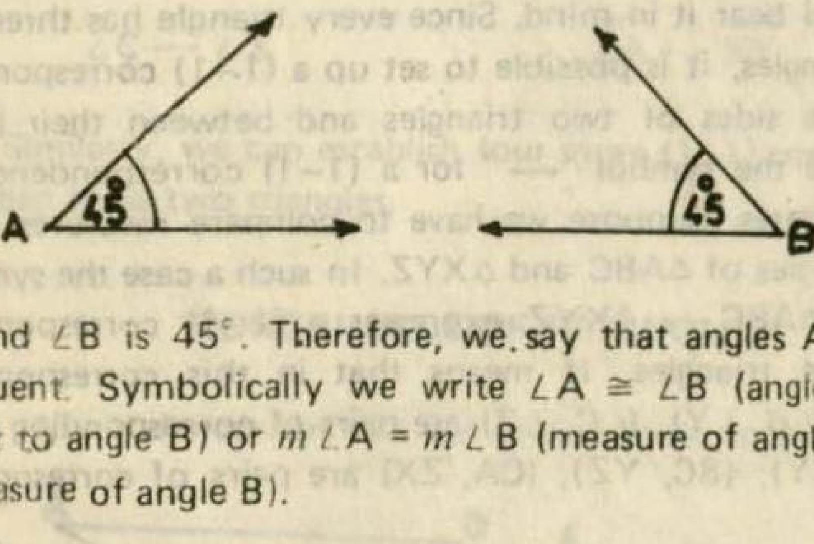
In such a case we say that $\overline{AB}$ is CONGRUENT to $\overline{CD}$ We can write this statement symbolically in one of the two ways.

(i) $\overline{AB} \cong \overline{CD}$ and read it as segment AB is congruent to segment CD.

(ii) AB = CD or $m\,\overline{AB} = m\,\overline{CD}$ and read it as " measure of segment AB is equal to measure of segment CD".

Similarly, if two angles have the same measure, they are called CONGRUENT ANGLES.

For example, in the following figure, measure of each of



$\angle A$ and $\angle B$ is 45°. Therefore, we say that angles A and B are congruent. Symbolically we write $\angle A \cong \angle B$ (angle A is congruent to angle B) or $m\,\angle A = m\,\angle B$ (measure of angle A is equal to measure of angle B).

To sum up:

If two line-segments are equal in measure (length), they are congruent segments and if two angles are equal in measure, they are congruent angles. We may also say:

(i) Every line-segment is congruent to itself and every angle is congruent to itself.

(ii) If two line-segments are congruent to the same line segment, they are congruent to each other.

(iii) If two angles are congruent to the same angle, they are congruent to each other.

(iv) All right angles are congruent.

(v) Complements of congruent angles are congruent.

(vi) Supplements of congruent angles are congruent.

(1–1) CORRESPONDENCE BETWEEN TWO TRIANGLES

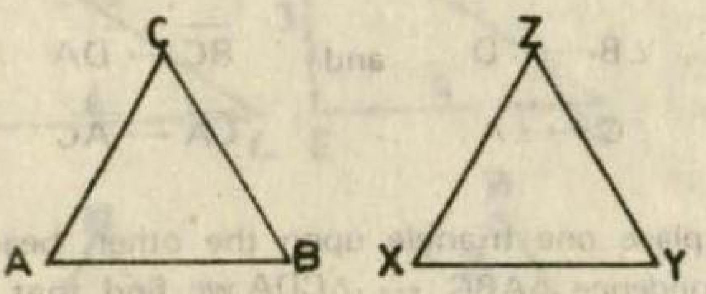
Sometimes we have to compare measures of the sides and angles of two triangles. In such a case it is quite essential to set up (1–1) correspondence between the sides and angles of the triangles and bear it in mind. Since every triangle has three sides and three angles, it is possible to set up a (1–1) correspondence between the sides of two triangles and between their angles. We shall use the symbol '$\longleftrightarrow$' for a (1–1) correspondence between the figures. Suppose we have to compare measures of the sides and angles of $\triangle ABC$ and $\triangle XYZ$. In such a case the symbolic expression $\triangle ABC \longleftrightarrow \triangle XYZ$ expresses a (1–1) correspondence between the triangles. It means that in this correspondence $(\angle A, \angle X)$, $(\angle B, \angle Y)$, $(\angle C, \angle Z)$ are pairs of corresponding angles and (AB, XY), (BC, YZ), (CA, ZX) are pairs of corresponding sides.

In other words, the symbolic expression
$\triangle ABC \longleftrightarrow \triangle XYZ$ means that

$\angle A \longleftrightarrow \angle X$ and $\overline{AB} \longleftrightarrow \overline{XY}$

$\angle B \longleftrightarrow \angle Y$ $\overline{BC} \longleftrightarrow \overline{YZ}$

$\angle C \longleftrightarrow \angle Z$ $\overline{CA} \longleftrightarrow \overline{ZX}$



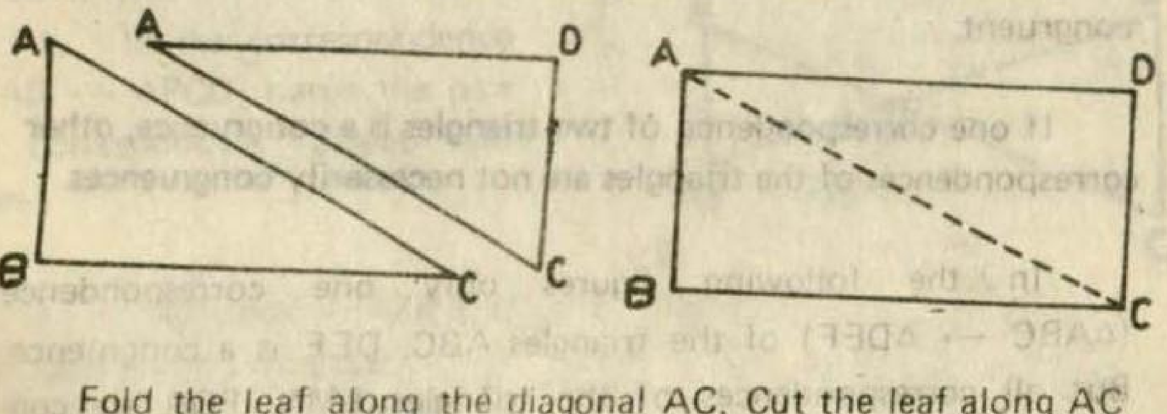
Another correspondence of the same triangles is represented by the expression $\triangle ABC \longleftrightarrow \triangle YZX$. This is equivalent to the following six relations:

$\angle A \longleftrightarrow \angle Y$	$\overline{AB} \longleftrightarrow \overline{YZ}$
$\angle B \longleftrightarrow \angle Z$	$\overline{BC} \longleftrightarrow \overline{ZX}$
$\angle C \longleftrightarrow \angle X$	$\overline{CA} \longleftrightarrow \overline{XY}$

Similarly, we can establish four more (1–1) correspondences between these two triangles.

CONGRUENCE OF TRIANGLES

Name the corners of a leaf of your note-book as A, B, C, D.



Fold the leaf along the diagonal AC. Cut the leaf along AC to obtain two triangular regions ABC, ADC. Are the boundaries of these regions alike triangles. If we consider the correspondence $\triangle ABC \longleftrightarrow \triangle CDA$ of the triangles, we shall have the following pairs of corresponding angles and sides.

$\angle A \leftrightarrow \angle C$ $\quad\quad$ $\overline{AB} \leftrightarrow \overline{CD}$

$\angle B \leftrightarrow \angle D$ $\quad$ and $\quad$ $\overline{BC} \leftrightarrow \overline{DA}$

$\angle C \leftrightarrow \angle A$ $\quad\quad$ $\overline{CA} \leftrightarrow \overline{AC}$

If we place one triangle upon the other, bearing in mind the correspondence $\triangle ABC \leftrightarrow \triangle CDA$,we find that their corresponding angles and sides are congruent and the triangles coincide with each other. Such two triangles are called CONGRUENT TRIANGLES. In such a case we also say that the correpondence $\triangle ABC \leftrightarrow \triangle CDA$ is a CONGRUENCE. This is symbolically expressed as

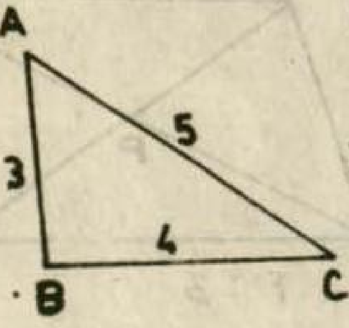
$$\triangle ABC \cong \triangle CDA$$

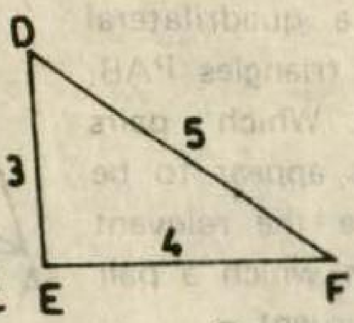
If it is possible to set up a (1–1) correspondence between two triangles, in which the corresponding sides and the corresponding angles are congruent, the triangles are said to be congruent.

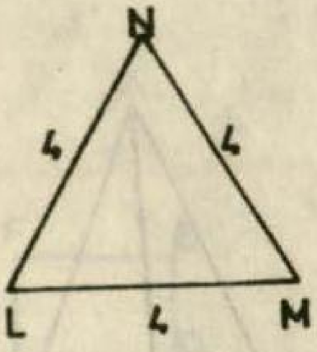
Similarly, if it is possible to set up a (1–1) correspondence between two polygons in which the corresponding sides and the corresponding angles are congruent, the polygons are said to be congruent.

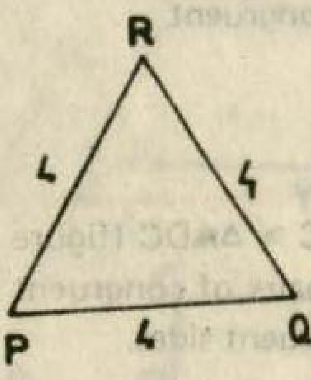
If one correspondence of two triangles is a congruence, other correspondences of the triangles are not necessarily congruences.

In the following figures only one correspondence ($\triangle ABC \leftrightarrow \triangle DEF$) of the triangles ABC, DEF is a congruence. But all correspondences of the triangles LMN, PQR are congruences.









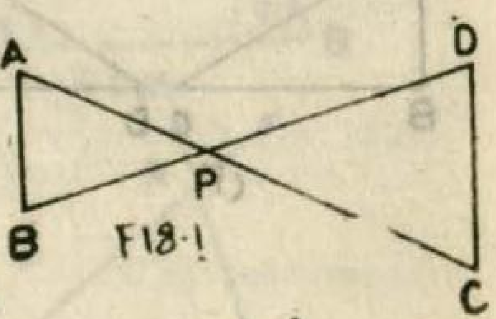
We may also say that

(i) Every triangle is congruent to itself (in the correspondence in which its sides and angles correspond to themselves). Such a congruence is called IDENTITY CONGRUENCE.

(ii) If each of two triangles is congruent to a third triangle, they are congruent to each other.

EXERCISE No. 10.7

1. In the correspondence $\triangle PAB \leftrightarrow \triangle PCD$, name the pair of corresponding angles and sides.



FIG·1

2. $\triangle ABC$ and $\triangle PQR$ are two equilateral triangles and AB = PQ. Which correspondences of the triangles are congruences ?

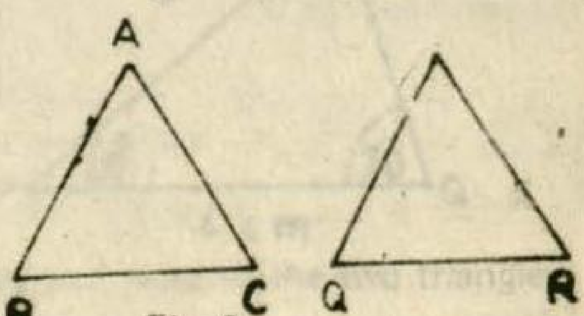


Fig. 2

3. In the adjoining figure (3) are shown a quadrilateral ABCD and four triangles PAB, PBC, PCD, PDA. Which pairs of these triangles appear to be congruent? Name the relevant correspondence in which a pair appears to be congruent.

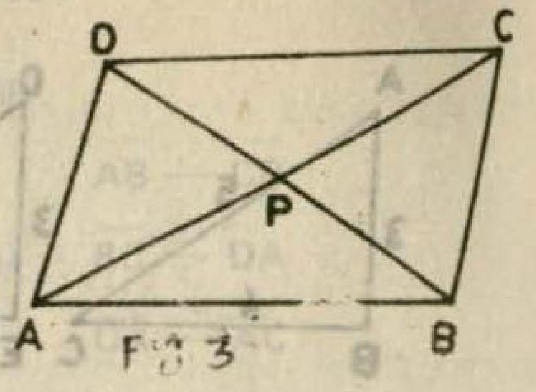


Fig 3

4. ΔABC ≅ ΔADC (figure 4). Name the pairs of congruent angles and congruent sides.

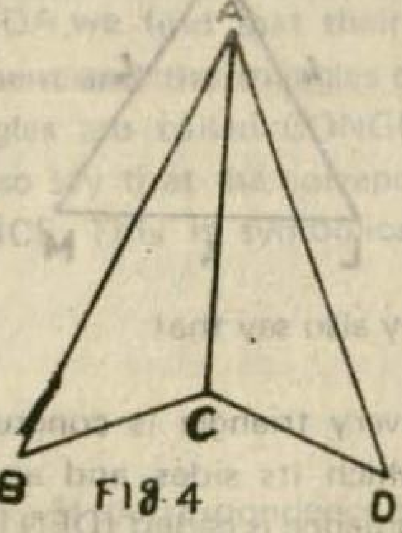
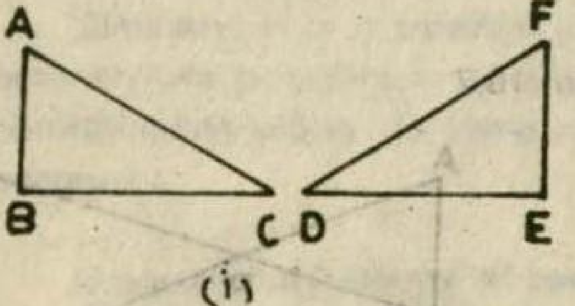
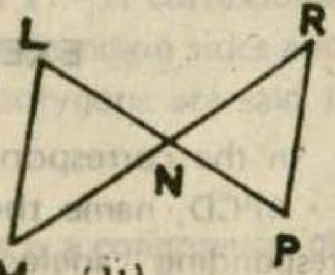


Fig 4

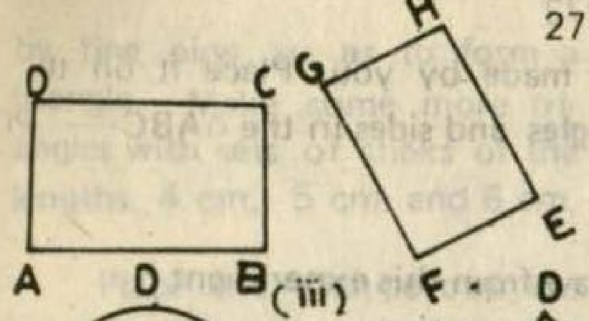
5. Below are given a number of pairs of figures. Which of the pairs of figures are congruent? Specify the relevant correspondences in case of congruent figures.



(i)



(ii)



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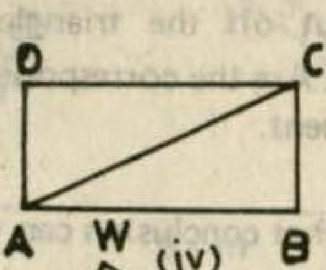
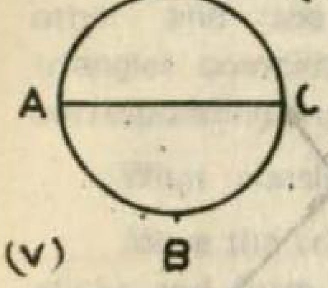
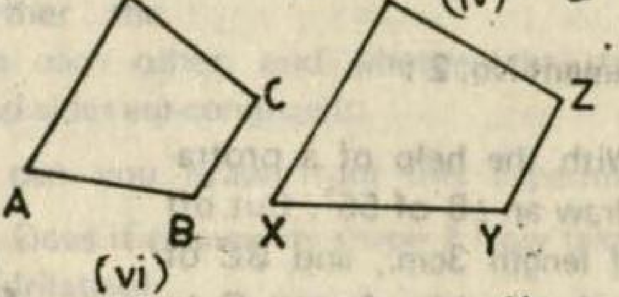


(iv)



(v)



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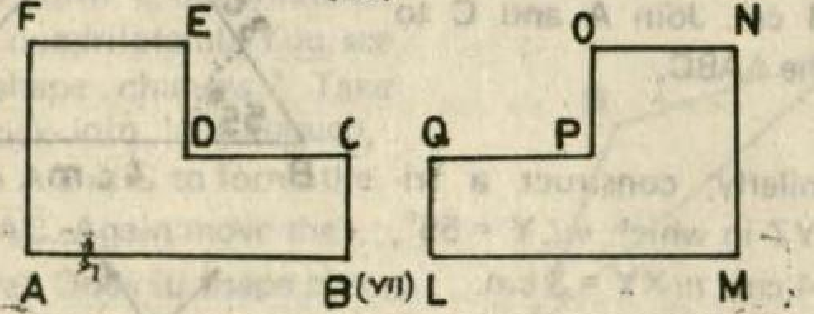
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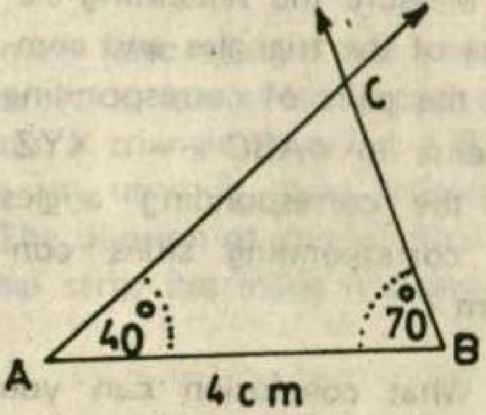
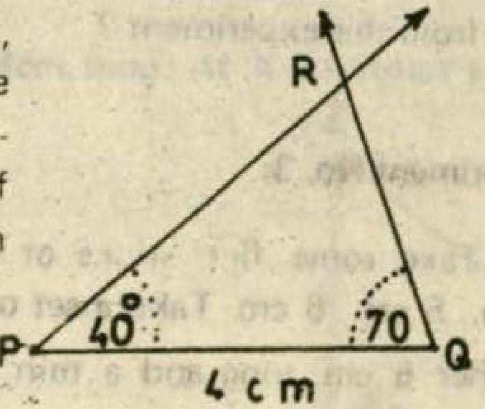
THEOREMS ABOUT CONGRUENT TRAINGLES


C
40°
70
A
B
4 cm

Experiment No. 1:

Draw a line-segment AB, 4cm. long. Construct an angle of 40° at A and an angle of 70° at B. Let C be the point of intersection of the non-common arms of the angles A, B.


R
40°
70
P
Q
4 cm

Draw a line-segment PQ, 4cm. long. At P and Q make angles of 40° and 70° respectively. Let R be the point of intersection of the non-common arms of the angles P, Q.

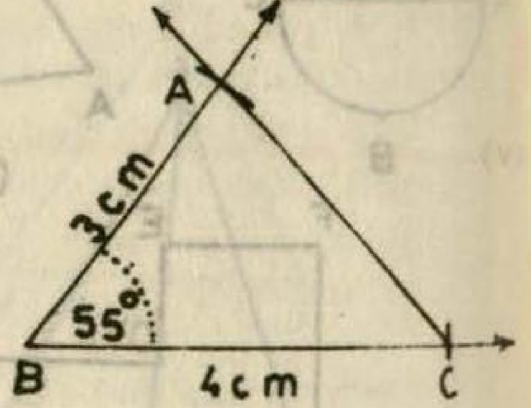
Measure the remaining angles and sides of the two triangles and compare the magnitudes of the corresponding angles and sides.

Cut off the triangle ABC made by you. Place it on the $\triangle PQR$. Are the corresponding angles and sides in the $\triangle ABC \leftrightarrow \triangle PQR$ congruent.

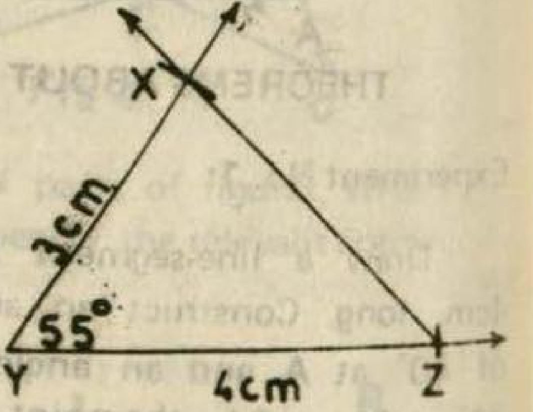
What conclusion can you draw from this experiment ?

Experiment No. 2 :

With the help of a protractor draw an $\angle B$ of 55°. Cut off $\overline{BA}$ of length 3cm., and $\overline{BC}$ of length 4 cm. Join A and C to obtain the $\triangle ABC$.



Similarly, construct a triangle XYZ in which $m\angle Y = 55^\circ$, $m\overline{YZ} = 4$ cm., $m\overline{XY} = 3$ cm.

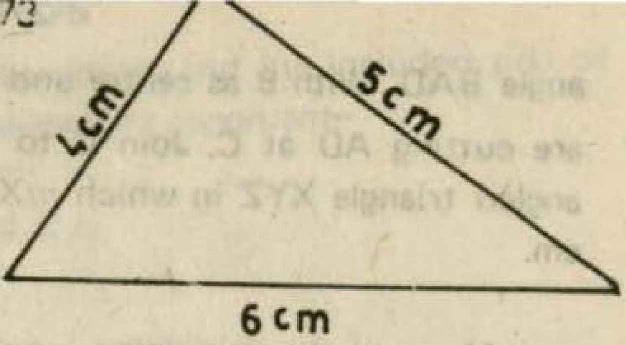


Measure the remaining elements of the triangles and compare the pairs of corresponding elements in $\triangle ABC \longleftrightarrow \triangle XYZ$. Are the corresponding angles and corresponding sides congruent ?

What conclusion can you draw from this experiment ?

Experiment No. 3:

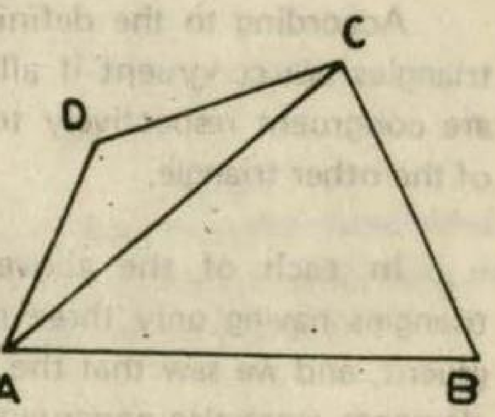
Take some fine sticks or strips of strawboard of lengths 4 cm., 5 cm., 6 cm Take a set of three of them, one 4 cm. long another 5 cm. long and a third 6 cm. long, and join their ends by fine pins, so as to form a triangle. Make some more triangles with sets of sticks of the lengths. 4 cm., 5 cm. and 6 cm.



Place one triangle over another and see whether the triangles coincide with each other, and whether the pairs of corresponding angles and sides are congruent.

What conslusion can you draw from this experiment ?

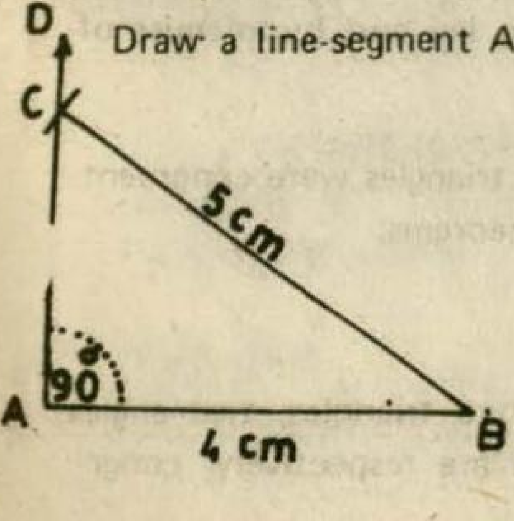
Move the triangle. Does it change its shape ? Now take four sticks and form a quadrilateral. Move the quadrilateral. You see that its shape changes ? Take anoher stick-join long enough, its ends to A and C to form the diagonal $\overline{AC}$. Again move the quadilateral. Does its hsape change now ?

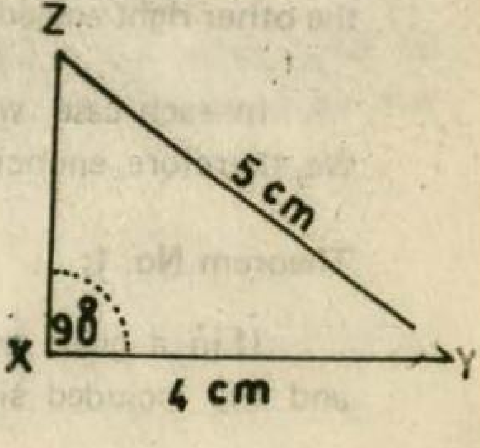


From the experiment you must have noticed that a triangle does not change its shape whereas a quadrilateral can easily change its shape. For this reason a triangle is called a RIGID FIGURE and triangular frames are used in most articles for making them strong and rigid. The division of quadrilatural into triangular frames by the diagonal strip has made it strong and rigid.

Experiment No. 4:

Draw a line-segment AB, 4cm, long. At A construct a right





angle BAD. With B as centre and radius of 5 cm. draw a circular are cutting $\overrightarrow{AD}$ at C. Join C to B. Similarly, construct a right-angled triangle XYZ in which $m\overline{XY} = 4$ cm, $m\angle x = 90^\circ$. $m\overline{YZ} = 5$ cm.

Measure the remaining angles and sides of both the triangles. In $\Delta ABC \longleftrightarrow \Delta XYZ$, are the corresponding angles and sides congruent ? What conclusion can you draw from this experiment?

Conclusions:

According to the definition of congruence of triangles two triangles are congruent if all the sides and angles of one triangle are congruent respectively to the corresponding sides and angles of the other triangle.

In each of the above experiments, we constructed two triangles having only three pairs of corresponding elements congruent, and we saw that the remaining pairs of the corresponding elements were also congruent. In Experiment No. 1 we constructed two triangles in which two angles and the included side of one were congruent to the corresponding angles and the included side of the other. In Experiment No. 2, two sides and the included angle of one triangle were made congruent to the corresponding sides and the included angle of the other triangle. In Experiment No. 3, all the three sides of one triangle were congruent to the corresponding sides of the other triangle. In Experiment No. 4, one leg and the hypotenuse of a right-angled triangle were constructed congruent to the corresponding leg and hypotenuse of the other right-angled triangle.

In each case we noticed that the triangles were congruent. We, therefore, enunciate the following theorems:

Theorem No. 1:

If in a given correspondence of two triangles, two angles and the included side of one triangle are respectively congruent to the corresponding two angles and the included side of the other triangle, then the triangles are congruent.

A.S.A. ≅ A.S.A.

Theorem No. 2:

If in a given correspondence of two triangles two sides and the included angle of one triangle are respectively congruent to the corresponding two sides and the included angle of the other triangle, then the triangles are congruent.

(S.A.S ≅ S.A.S)

Theorem No. 3:

If in a given correspondence of two triangles three sides of one triangle are respectively congruent to the corresponding sides of the other triangle, then the triangles are congruent.

(S.S.S ≅ S.S.S)

Theorem No. 4:

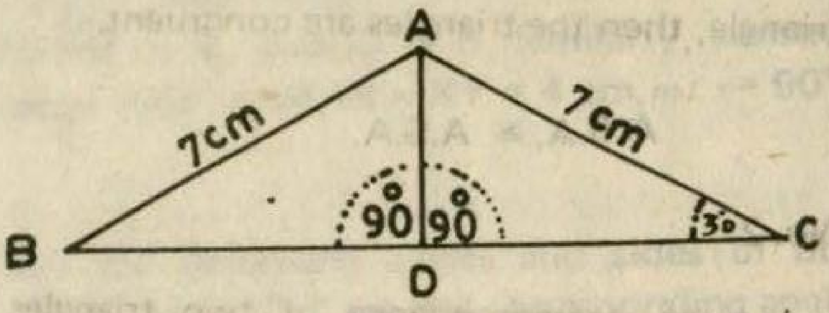
If in a given correspondence of two right angled triangles one leg and the hypotenuse of one triangle are respectively congruent to the corresponding leg and the hypotenuse of the other triangle, then the triangles are congruent.

(HL ≅ HL)

The congruence theorems of triangles are very useful in solving many geometrical problems. This is illustrated by the following examples :

Example 1

In the following figure we want to find the measure of



angle B.

In the correspondence of the right-angled triangles

$$\triangle ADB \leftrightarrow \triangle ADC,$$

$$\therefore \begin{cases} \text{Hypotenuse } \overline{AB} \cong \text{hypotenuse } \overline{AC} \\ \text{leg } \overline{AD} \cong \text{leg } \overline{AD} \end{cases}$$

$\therefore \quad \triangle ADB \cong \triangle ADC \qquad (H.L \cong H.L)$

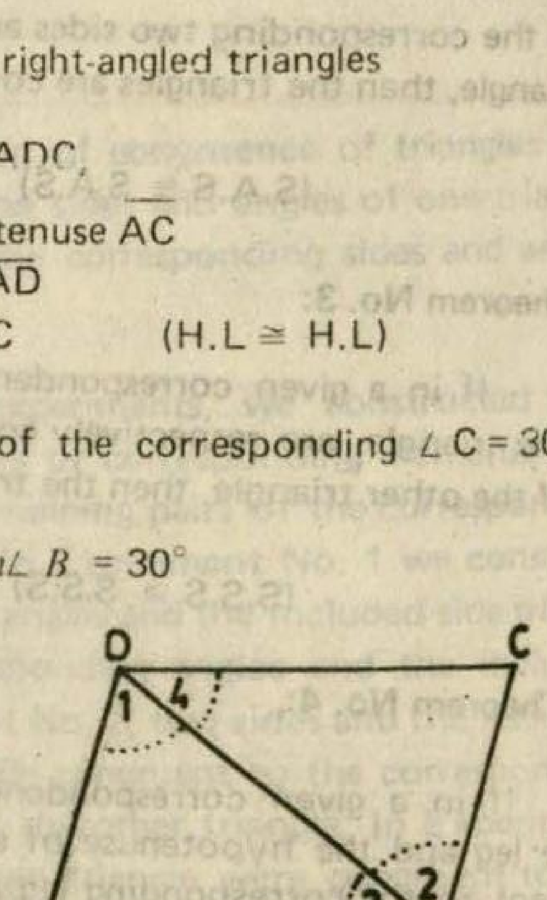
Therefore, $m\angle B$ = measure of the corresponding $\angle C = 30^\circ$ of the other triangle.

$$\therefore \quad m\angle B = 30^\circ$$

Example 2

Prove that the diagonal of a parallelogram divides it into two congruent triangles.

This theorem may be proved as follows :



Given:

Parallelogram ABCD in which $\overline{BD}$ is a diagonal.

To prove :

$\triangle DAB \cong \triangle BCD$

Proof:

In the correspondence $\triangle DAB \leftrightarrow \triangle BCD$

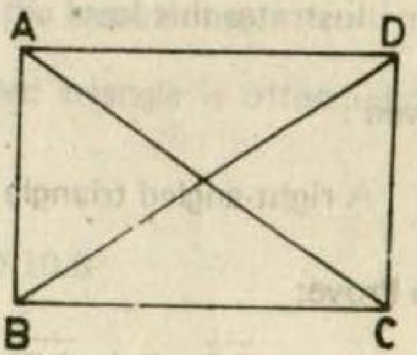
$$\because \begin{cases} m\overline{DB} = m\overline{DB}, & \text{(Common)} \\ m\angle 1 = m\angle 2, & \text{(Alt. } \angle\text{s of } \parallel \text{ lines)} \\ m\angle 3 = m\angle 4 & \text{(Alt. } \angle\text{s of } \parallel \text{ lines)} \end{cases}$$

$\therefore \triangle DAB \cong \triangle BCD \quad (A{\cdot}S{\cdot}A \cong A{\cdot}S{\cdot}A)$

Example 3

Prove that the diagonals of a rectangle are congruent.

We prove this theorem by constructing a rectangle as shown in the adjoining figure.



Given:;

A rectangle ABCD in which $\overline{AC}$ and $\overline{BD}$ are its diagonals.

To Prove:

$m\overline{AC} = m\overline{BD}$

Proof:

In the correspondence $\angle ABC \leftrightarrow \triangle DCB$

$$\because \begin{cases} \overline{BC} = \overline{BC}, & \text{(common)} \\ \overline{AB} \cong \overline{DC}, & \text{(opposite sides of the rectangle*)} \\ m\angle ABC = m\angle DCB & \text{(right } \angle\text{ s)} \end{cases}$$

$\therefore \quad \triangle ABC \cong \triangle DCB \quad (S.A.S \cong S.A.S.)$

$\therefore \quad \overline{AC} \cong \overline{BD}$

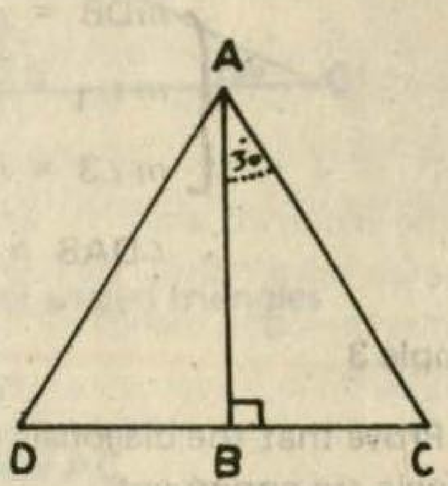
---

* In a rectangle all the angles are right angles and opposite sides are congruent.

Example 4

Prove that in a right-angled triangle with one acute angle of 30°, the side opposite that angle is half as long as the hypotenuse.

Note:- Sometimes we make an addition to the given figure for proving the theorem. The proof of the above theorem illustrates this fact.



Given :

A right-angled triangle ABC in which $m\angle BAC = 30^\circ$.

To Prove:

$$m\overline{BC} = \tfrac{1}{2}(m\overline{AC})$$

Construction:

Produce $\overline{CB}$ beyond **B** and cut off BD congruent to BC. Join D to A.

Proof:

$m\angle ABC + m\angle ABD = 180^\circ$, (Adjacent supplementary angles)

$\therefore$ $m\angle ABD = 90^\circ$ (Supplement of a right angle)

In $\triangle ABC \longleftrightarrow \triangle ABD$

$$\because \left\{ \begin{array}{l} m\overline{AB} = m\overline{AB}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(Common)} \\ m\overline{BC} = m\overline{BD} \text{ (Construction)} \\ \angle ABC = \angle ABD \text{ (Right angles)} \end{array} \right.$$

$\triangle ABC \cong \triangle ABD$ (S.A.S. ≅ S.A.S.)

$\therefore \quad m\angle BAD = m\angle BAC = 30^\circ$

$m\angle DAC = 30^\circ + 30^\circ = 60^\circ$

Now in the triangle ADC all the angles are 60° each.

$\therefore \quad \triangle ADC$ is equilateral.

$\therefore \quad \overline{DC} \cong \overline{AC}$

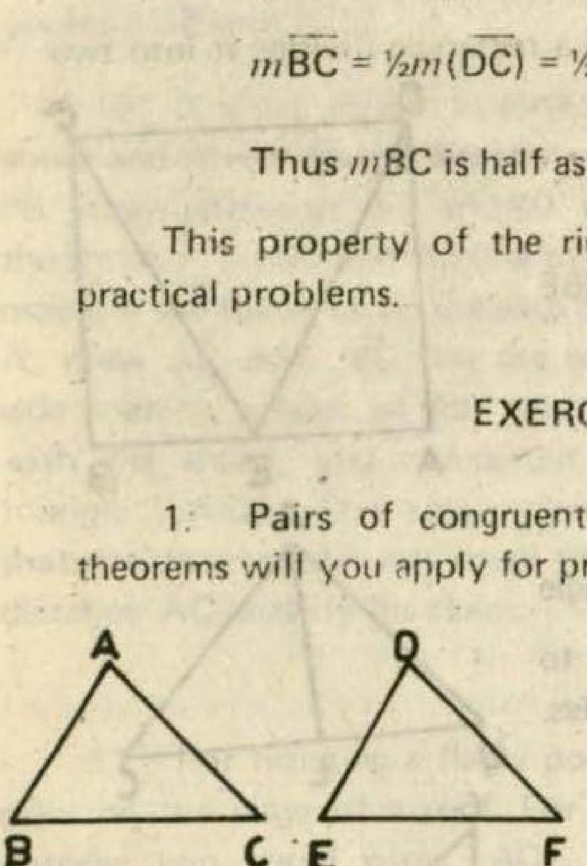
$m\overline{BC} = \tfrac{1}{2} m(\overline{DC}) = \tfrac{1}{2} m(\overline{AC})$

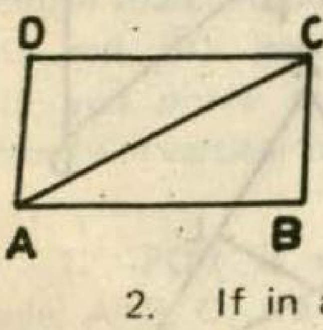
Thus $m\overline{BC}$ is half as long as the hypotenuse.

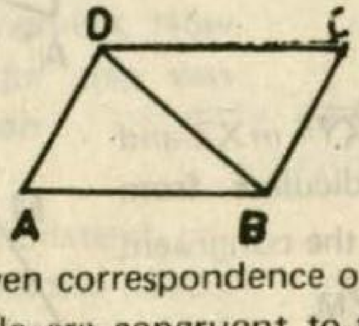
This property of the right angled triangle is often used in practical problems.

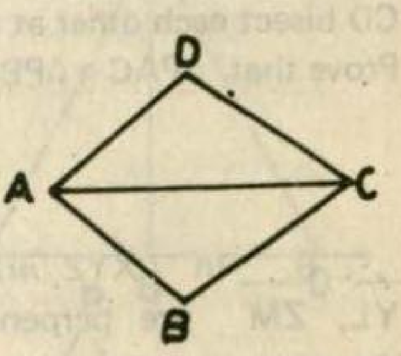
EXERCISE NO.10.8

1. Pairs of congruent triangles are given below. Which theorems will you apply for proving their congruence:









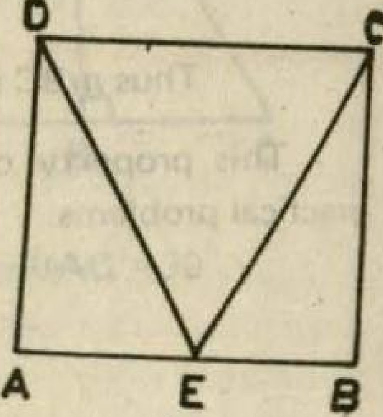
2. If in a given correspondence of two triangles the three angles of one triangle are congruent to the corresponding angles of the other, will the triangles be congruent ? Verify by drawing a figure.

3. If in a given correspondence of two triangles two angles and a non-included side of one triangle are congruent to the corresponding two angles and a non-included side of the other triangle, will the triangles be congruent ?

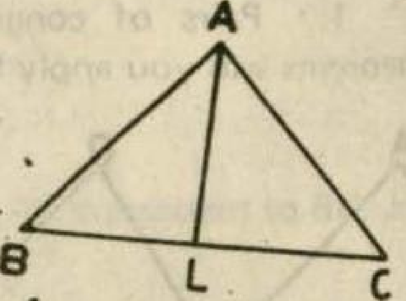
(What about the magnitudes of the third angle of each of the triangles ? What form does the theorem take now)?

4. Prove that a diagonal of a rectangle divides it into two congruent triangles.

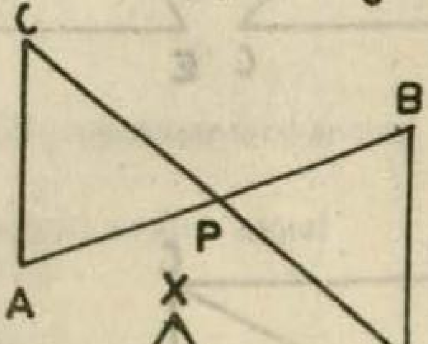
5. In the adjoining figure is shown a square ABCD. Prove that the correspondence $\triangle DAE \longleftrightarrow \triangle CBE$ is a congruence.



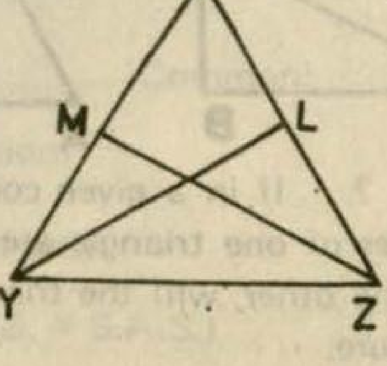
6. The bisector $\overline{AL}$ of angle A of $\triangle ABC$ is perpendicular to $\overline{BC}$. Prove that the triangle is isosceles.



7. Line-segments AB and CD bisect each other at a point P. Prove that $\triangle PAC \cong \triangle PBD$.

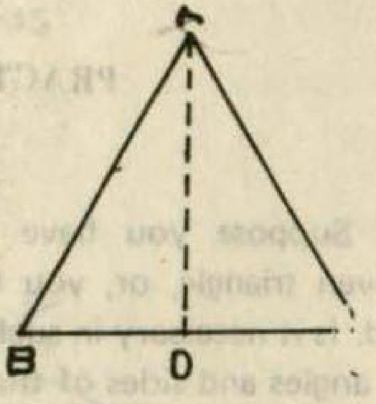


8. In $\triangle XYZ$, $m\overline{XY} = m\overline{XZ}$ and YL, ZM are perpendiculars from the ends of the base to the congruent sides. Prove that $\overline{YL} \cong \overline{ZM}$.

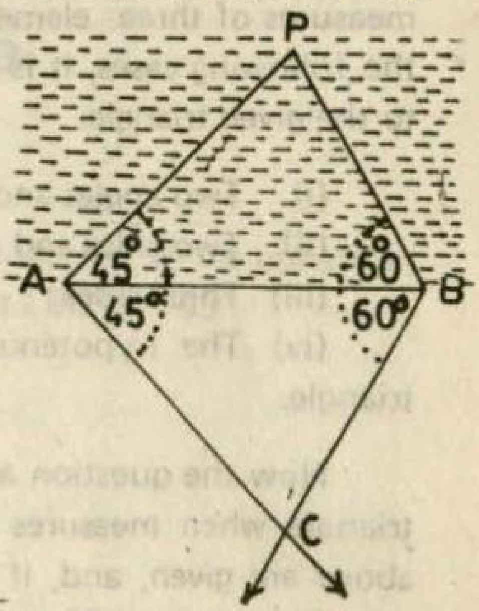


9. In $\triangle ABC$, $\overline{AB} \cong \overline{AC}$. Prove that $\angle B \cong \angle C$.

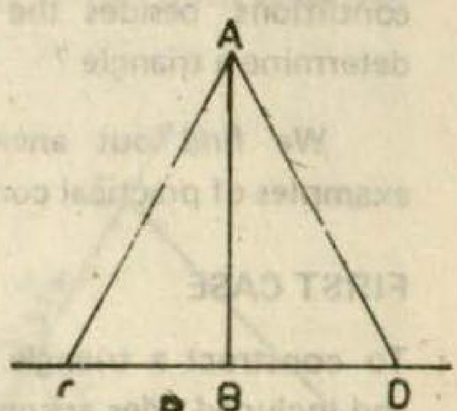
(*Hint* : Let $\overrightarrow{AD}$ be the bisector of angle A. Derive the requisite result considering the congruence of triangles ADB and ADC.



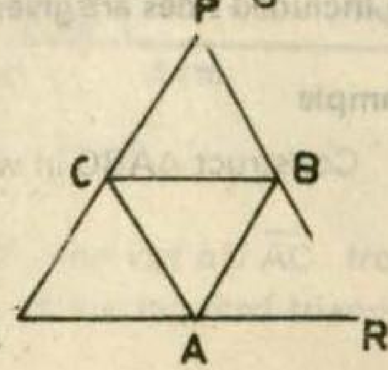
10. A ship P was observed from two points A, B on the shore and it was found that $\overrightarrow{PA}$ and $\overrightarrow{PB}$ make angles of 45° and 60° with the shore. A scout who wanted to measure the distance of the ship form A, drew $\overrightarrow{AC}$ and $\overrightarrow{BC}$ on the other side making angles of 45° and 60° with the shore, and completed the triangle ACB. The scout claimed that the distance AP was equal to the distance AC. Justify his claim.



11. For hoisting a flag a pole $\overline{AB}$ had to be erected vertically on the edge of a roof. For this purpose two equal wires $\overline{AC}$, $\overline{AD}$ were attached to A and fixed at points C, D so that the wires were in stretched position and $\overline{BC}$ and $\overline{BD}$ were congruent. How will you prove that the pole was actually in vertical position ?



12. PQR is an equilateral triangle. A, B, C are the middle points of its sides. Prove that $\triangle ABC$ is also equilateral.



CHAPTER 11

PRACTICAL GEOMETRY

Suppose you have to construct a triangle congruent to a given triangle, or, you have to draw the sketch of a triangular field. Is it necessary in such cases to know the measures of all the angles and sides of the triangles ? In the light of the theorems relating to congruence of triangles, you can say that having known measures of three elements of the triangle as given in any one of the following cases, it is possible to construct a triangle congruent to the given triangle.

(i) Two angles and the included side.

(ii) Two sides and the included angle.

(iii) Three sides.

(iv) The hypotenuse and a leg in case of a right-angled triangle.

Now the question arises: Is it always possible to construct a triangle when measures of its elements mentioned in (i) to (iv) above are given, and, if so, how many triangles will satisfy each set of conditions. Moreover, is there any other set of minimum conditions, besides the four sets mentioned above, which will determine a triangle ?

We find out answers to these questions by considering examples of practical constructions.

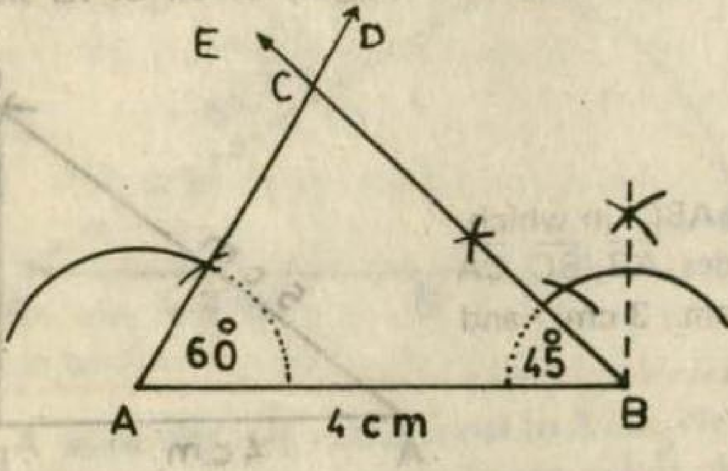
FIRST CASE

To construct a triangle when the measures of two of its angles and included sides are given.

Example

Construct $\triangle ABC$ in which

$\overline{AB}$ = 4cm., $m\angle A$ =60°, $m\angle B$=45°



Construction

We draw a line-segment AB, 4 cm. long. At A, we construct $\angle DAB$ of 60°. Similarly, at B we construct $\angle EBA$ of 45°.

$\overrightarrow{BE}$ and $\overrightarrow{AD}$ intersect in C.

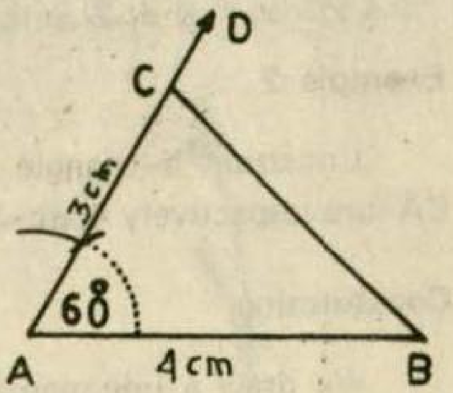
$\triangle$ ABC is the required triangle.

SECOND CASE

To construct a triangle when the measures of two of its sides and the included angle are given.

Example

Construct a $\triangle ABC$ in which measures of $\overline{AB}$, $\overline{AC}$ and $\angle A$ are respectively 4 cm., 3cm.,and 60°.



Construction

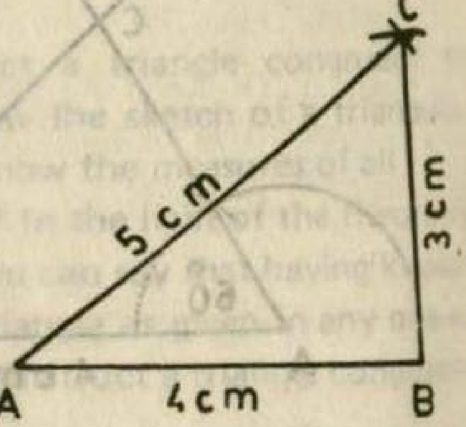
We draw a line-segment AB, 4 cm., long.

At A we construct $\angle DAB$ of 60°, and cut off $\overline{AC}$ from $\overrightarrow{AD}$ 3cm. long. We join C to B. $\triangle ABC$ is the required triangle.

THIRD CASE

To construct a triangle when the lengths of all of its sides are given.

Example 1



Construct a △ABC in which lengths of the sides $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ are respectively 4cm., 3 cm., and 5 cm.

Construction

We draw a segment AB, 4 cm. long. With A as centre and radius equal to 5 cm. we draw an arc. Similarly, we draw another arc with B as centre and radius equal to 3cm. The two arcs intersect at a point C. We joint C to A and B.

△ABC is the required triangle.

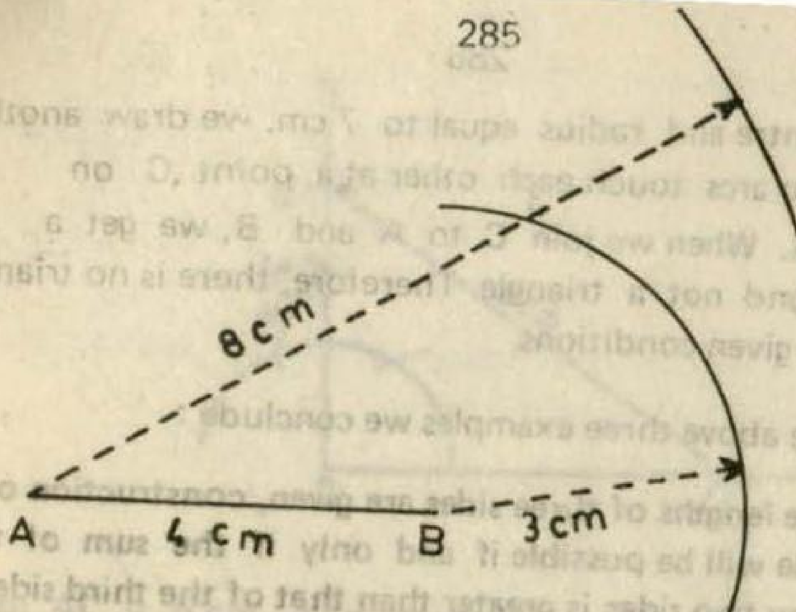
N.B. : It may be noted that in the data of this example the sum of the lengths of any two sides of the required triangle is greater than that of the third side and we get a unique triangle fulfilling the given conditions.

Example 2

Construct a triangle ABC in which lengths of $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ are respectively 4cm., 3cm., and 8 cm.

Consturction

We draw a line-segment AB, 4cm. long. With B as centre and radius equal to 3cm. we draw an arc.



With A as centre and radius equal to 8 cm. We draw another arc. These two arcs do not intersect in any point. We thus fail to find the position of the vertex C. Therefore, the data in this case do not determine any triangle.

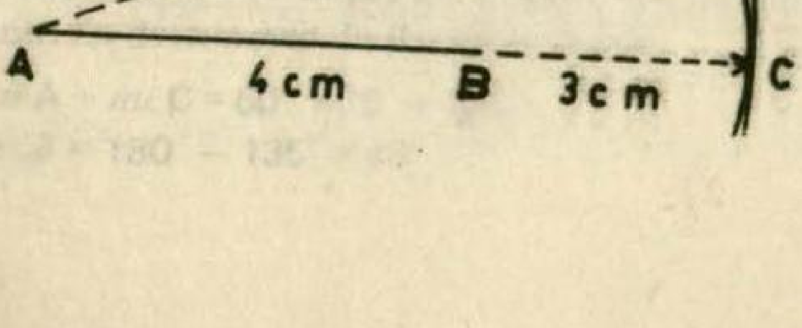
N.B. : It may be noted that in Example 2, the sum of the lengths of the first two sides is less than that of the third side.

Example 3

Construct a triangle ABC in which lengths of the sides $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ are 4 cm., 3 cm., 7 cm., respectivly.

Construction

We draw a line-segment AB, 4cm. long. With B as centre and radius equal to 3cm. we draw an arc.



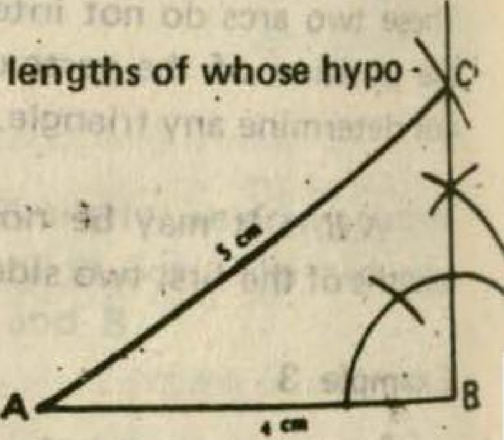
With A as centre and radius equal to 7 cm. we draw another arc. These two arcs touch each other at a point C on $\overrightarrow{AB}$ beyond A. When we join C to A and B, we get a line-segment AC and not a triangle. Therefore, there is no triangle satisfying the given conditions.

From the above three examples we conclude :

When the lengths of three sides are given, construction of a unique triangle will be possible if and only if the sum of the lengths of any two sides is greater than that of the third side.

FOURTH CASE

To construct a right-angled triangle the lengths of whose hypotenuse and one leg are known.



Example 1

Constuct a rightangled triangle ABC in which hypotenuse AC is 5 cm. long and one leg $\overline{AB}$ is 4 cm. long.

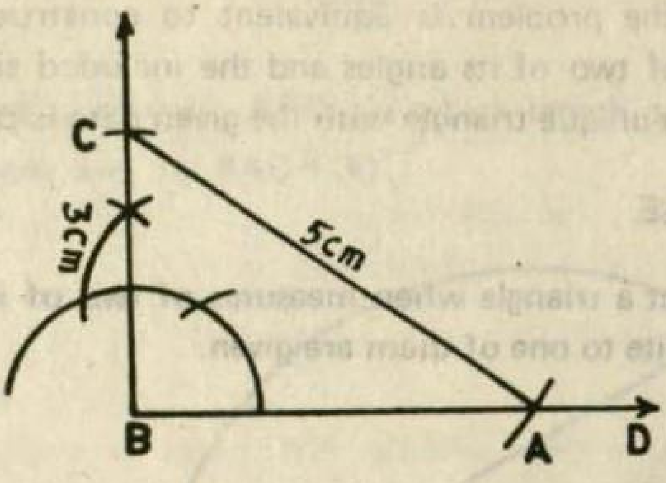
Construction

We draw $\overline{AB}$, 4 cm. long. At B we construct an angle ABC of 90°. With A as centre and radius equal to 5 cm. we draw an arc cutting $\overline{BC}$ in C. We join C to A.

△ ABC is the required triangle.

Example 2

Construct a right-angled triangle ABC in which the hypo-



tenuse AC is 5 cm. long and one leg BC is 3 cm. long.

Construction

We draw a ray BD. At B we construct a right angle DBC.

From $\overrightarrow{BC}$ we cut off $\overline{BC}$ 3 cm. long With C as centre and radius equal to 5cm. we draw an arc cutting $\overrightarrow{BD}$ in A. We join A to C. $\triangle ABC$ is the required triangle.

FIFTH CASE

To construct a triangle when measures of two of its angles and a side opposite to one of them are given.

Example

Construct a triangle ABC in which

$$m\angle A = 60^\circ,\ m\angle C = 75^\circ \text{ and } m\overline{AB} = 3\text{cm.}$$

Guidelines for construction :

Hint : You have learnt in the previous class that the sum of the measures of the three angles of a triangle is 180° . Therefore, when the measures of two angles are known, the measure of the third angle can be determined. In the present case

$$m\angle A + m\angle C = 60^\circ + 75^\circ = 135^\circ$$
$$\therefore\ m\angle B = 180^\circ - 135^\circ = 45^\circ$$

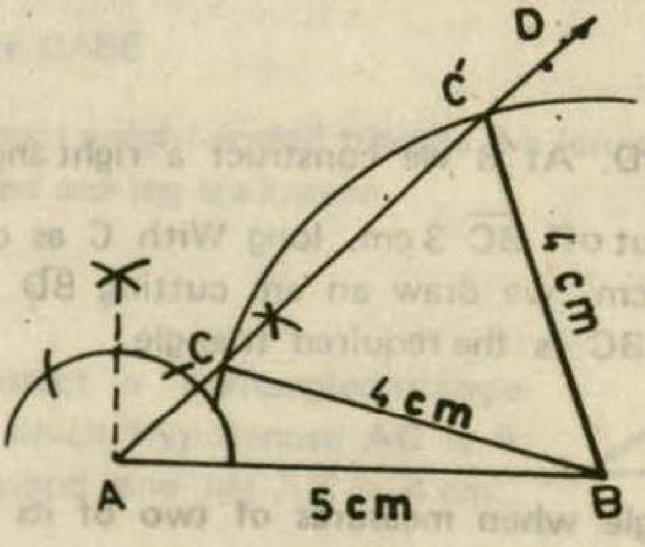
Now the problem is equivalent to constructing a triangle mreasures of two of its angles and the included side being given. Therefore, a unique triangle with the given data is possible.

SIXTH CASE

To construct a triangle when measures of two of its sides and the angle opposite to one of them are given.

Example 1

Construct $\triangle ABC$ in which lengths of $\overline{AB}$ and $\overline{BC}$ are 5cm., and 4 cm., respectively and $m\angle CAB = 45^\circ$



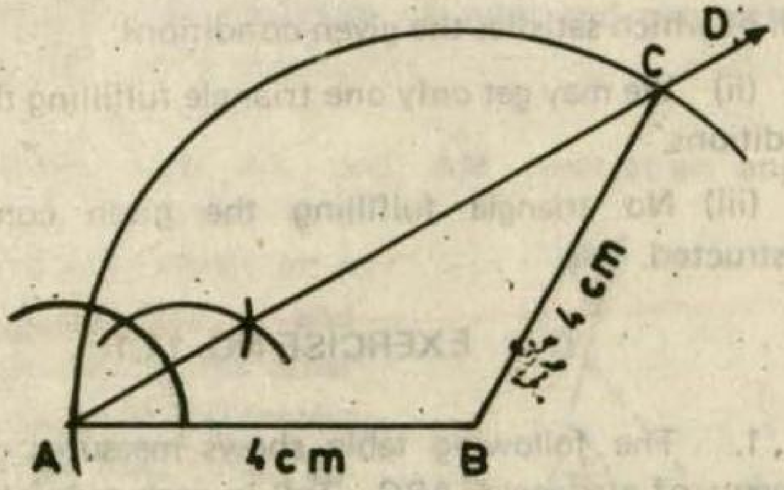
Construction

We draw a line-segment AB, 5 cm. long. At A we construct angle DAB of 45°. With B as centre and radius equal to 4 cm. we draw an arc cutting $\overrightarrow{AD}$ in two points C, C'. We join C and C' to B.

Both the triangles ABC and ABC' are the required triangles as each one of them satisfies the given conditions.

Example 2

Construct a triangle ABC in which length of each of $\overline{AB}$ and $\overline{BC}$ is 4cm. and $m\angle BAC = 30^\circ$.



Construction

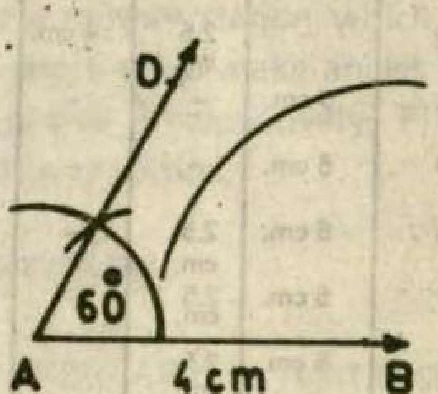
We draw $\overline{AB}$, 4cm. long. At A we construct $\angle DAB$ of 30°. With B as centre and radius equal to 4 cm. we draw an arc which cuts $\overrightarrow{AD}$ in A and C. We join C to B.

$\triangle ABC$ is the only triangle which satisfies the given conditions.

Example 3

Construct a triangle ABC in which $\overline{AB}$ and $\overline{BC}$ have respectively the lengths 4cm. and 2.5 cm. and $m\angle CAB = 60^\circ$.

Construction



We draw $\overline{AB}$, 4cm. long. At A we draw $\angle BAD$ of 60°. With B as centre and radius equal to 2.5 cm. we draw an arc which fails to meet $\overrightarrow{AD}$ in any point.

Therefore, there is no triangle satisfying the given conditions.

From the above three examples we conclude:

When measures of two sides and non-included angle of a triangle are given, the following three possibilities can arise:

(i) We may be able to construct two different triangles each of which satisfies the given conditions;

(ii) We may get only one triangle fulfilling the given conditions.

(iii) No triangle fulfilling the given conditions can be constructed.

EXERCISE NO. 11.1

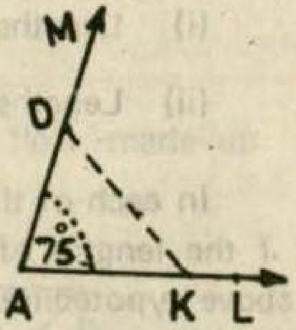
1. The following table shows measures of some of the elements of a triangle ABC. Tell in each case whether construction of the triangle is possible and if so, how many triangles you will get. (If necessary, you may draw figures in any case to find out the correct answers :

	$m\overline{AB}$	$m\overline{BC}$	$m\overline{CA}$	$m\angle A$	$m\angle B$	$m\angle C$	No. of possible different triangles
i	5 cm.	2.5 cm.	4 cm	–	–	–	
ii	–	–	–	30°	50°	100°	
iii	6 cm.	–	–	30°	50°	–	
iv	5 cm.	–	4 cm.	60°	–	–	
v	–	2.5 cm.	4 cm.	–	–	100°	
vi	5 cm.	–	–	30°	50°	–	
vii	5 cm.	–	–	–	50°	100°	
viii	5 cm.	2.5 cm.	–	–	145°	–	
ix	5 cm.	2.5 cm.	–	60°	–	–	
x	5 cm.	2.5 cm.	–	–	–	25°	
xi	7 cm.	2.5 cm.	4 cm.	–	–	–	
xii	6.5cm.	2.5 cm.	4 cm.	–	–	–	

2. Plan of a house which is to be built on a triangular piece of land has to be prepared. Measures of at least how many and of which angles and sides should be known for this purpose ?

3. Verify your answers to Q.1, parts (i) to (xii) by constructing the triangles with the help of ruler and protractor. State the steps of construction also.

4. Two straight roads $\overrightarrow{AL}$ and $\overrightarrow{AM}$ meet at an angle of 75° at a point A. The school K is situated on one of the roads at a distance of 600 meters from A and a bus-stop is situated on the other road at D at a distance of 450 meters from A. A kacha road joins the school and the bus-stop. Draw a sketch of the three places and find out how far is the school from the bus-stop ?



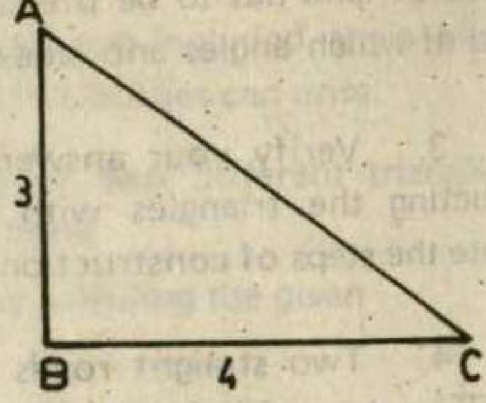
5. The TV station is at a distance of 10 km., from Abid's house and 8 km. from the school towards its north. Abid's house is towards east of the school. Draw a sketch of the three places. Measure the angle that the line through the house and T.V. station makes with the line through the house and the school.

6. Two villages A, B, are situated besides a long straight road and are 6 km. apart. The road goes from north to south Two short roads join the villages to a railway station which is towards west of the main road. The short roads make angles of 45° and 60° with the main road at A and B respectively. Find the distances of the villages from the railway station.

PYTHAGORAS THEOREM

1. We construct a $\triangle ABC$ in which $\angle B$ is a right angle, lengths of $\overline{AB}$, $\overline{BC}$ are 3 and 4 units respectively. On

measuring we find the length of $\overline{AC}$ to be 5 units.



Construct the following right-angled triangles and measure the length of the hypotenuse in each case:

(i) Lengths of the two legs are 6cm., 8 cm.;

(ii) Lengths of the two legs are 12 cm., 5 cm.;

In each of the above three cases compare sum of the squares of the lengths of the legs with the square of the length of the above hypotenuse.

What conclusion do you draw from the comparison ?

2. Below are given three sets of the measures of the sides of triangles. Construct the triangles and measure the greatest angle of the triangle in each case.

[7.5, 6, 4.5], [9, 12, 15], [6.5, 6, 2.5]

Compare sum of the squares of the lengths of two shorter sides with the square of the length of the longest side.

Do you notice that if the sum of the squares of the lengths of two sides of a triangle is equal to the square of the length of the third side, the triangle is right-angled?

THEOREM

We are now in a position to state that:

(i) In a right-angled triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of its two legs.

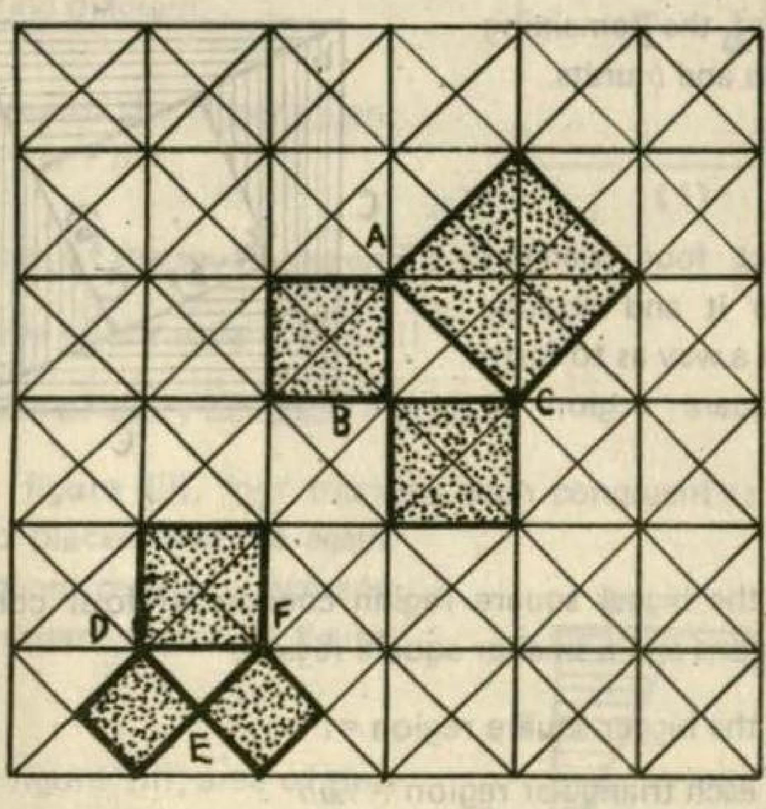
(ii) If in a triangle, the square of the length of a side is equal to the sum of the squares of the lengths of the other two sides, the angle opposite to the longest side is a right angle.

The proof of this theorem was discovered, about 2500 years ago, by Pythagoras, a well-known mathematician, and the theorem has been named after him.

We give below some informal proofs of the theorem.

Proof No. 1

In the following figure has been shown a floor made up of congruent isosceles right-angled triangles.



Square regions have been constructed on the sides of △ABC You have noticed that all the triangles are congruent.

(i) How many triangular regions are equal to the largest square region ?

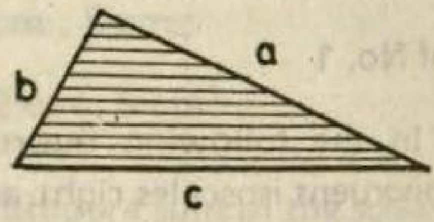
(ii) Find out the areas of the remaining two square regions also.

(iii) Is the area of the square region on $\overline{AC}$ equal to the sum of the areas of the square regions on $\overline{AB}$ and $\overline{BC}$.

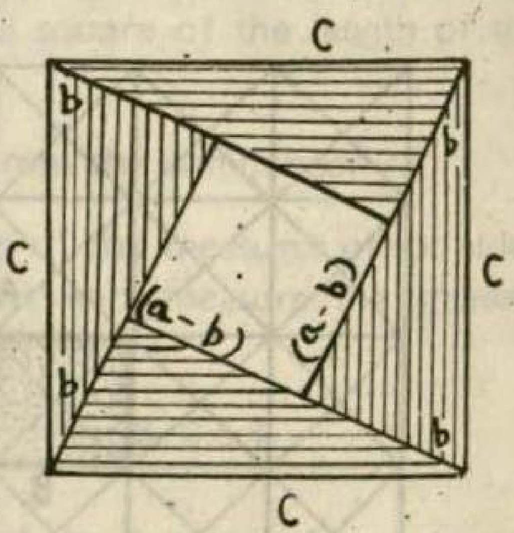
Can you draw the conclusion that :

"In a right-angled triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two legs."

Proof No.2



Suppose that in a right angled triangle the length of the hypotenuse is c units and the lengths of the remaining two sides are a and b units.



Construct four triangles congruent to it and arrange them in such a way as to form a bigger square region as in the figure.

In fact the bigger square region consists of four congruent triangular regions and a smaller square region.

Area of the bigger square region $= c^2$

Area of each triangular region $= \frac{1}{2}ab$

Area of the four triangular regions $= 4.\ \frac{1}{2}ab = 2ab$

Area of the smaller square region $= (a-b)^2$

But area of the bigger square region = area of the four Δr regions + area of the smaller square region.

$$\therefore \quad c^2 = 2ab + (a-b)^2$$

or $$c^2 = a^2 + b^2$$

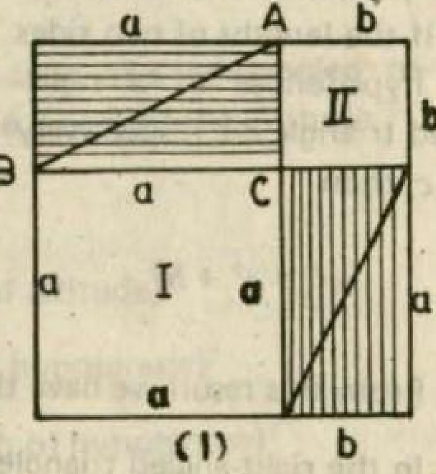
This proves the theorem.

Proof No.3

Suppose length of the hypotenuse of a right-angled triangle ABC is c units and lengths of the remaining two sides are a, b, units.

In figure (i) we have constructed square regions on the base and the altitude of the triangle and completed the remaining diagram.

Area of the bigger square region = $(a + b)^2$



(I)

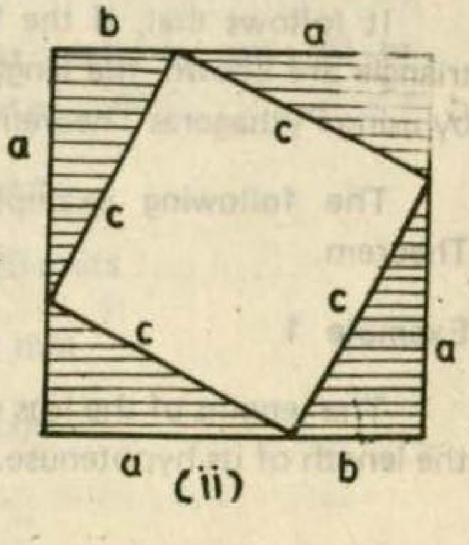
Area of the square region I = a^2

Area of the square region II = b^2

Area of every $\triangle r$ region = $\frac{1}{2}ab$

In figure (ii), four triangles each congruent to $\triangle$ABC have been so placed that we again get a square region, congruent to the square region in figure (i).

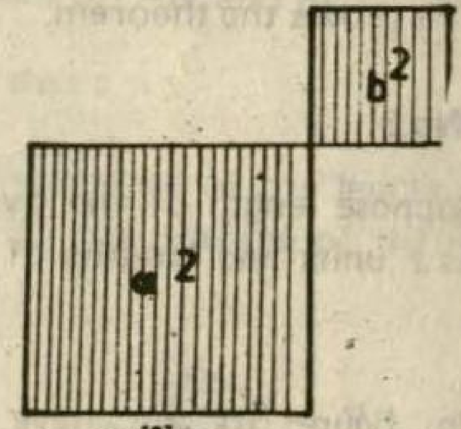
In figure (ii), area of the smaller square region is c^2. If all the triangular regions are deleted from figure (i), the area of the remaining region = $a^2 + b^2$ (figure iii).



(ii)

Similarly, if all the triangular regions are deleted from figure (ii), the area of the remaining region = c^2

Hence $c^2 = a^2 + b^2$



(iii)

In each of the above three proofs we have ~~verified~~ that if the lengths of two sides and hypotenuse of a right-angled triangle are respectively a, b, c, then

$$c^2 = a^2 + b^2$$

From this result we have the immediate deductions:

In the right-angled triangle ABC,

$$\because \quad c^2 = a^2 + b^2 \qquad \text{[Pythagoras theorem]}$$

$$\therefore \quad a^2 = c^2 - b^2 \qquad \ldots \quad \text{(i)}$$

$$\text{and} \quad b^2 = c^2 - a^2 \qquad \ldots \quad \text{(ii)}$$

It follows that, if the lengths of two sides of a right-angled triangle are known, the length of the third side can be determined by using Pythagoras Theorem.

The following examples illustrate the use of Pythagoras Theorem.

Example 1

The lengths of the legs of a right-angled triangle are a, b. Find the length of its hypotenuse.

Solution

(Length of hypotenuse)^2 = Sum of squares of the lengths of the legs.

$$= a^2 + b^2$$

Length of hypotenuse $= \sqrt{a^2 + b^2}$

Examples 2

The lengths of the hypotenuse and base of a right-angled triangle are c, a respectively. What is the length of the altitude ?

Solution

$\because$ (Length of base)^2 + (length of altitude)^2

= (length of hypotenuse)^2

$\therefore$ (length of altitude)^2 = (length of hypotenuse)^2 − (length of base)^2

$$= c^2 - a^2$$

$\therefore$ length of altitude $= \sqrt{c^2 - a^2}$

EXERCISE NO. 11.2

1. In a triangle ABC, $\angle C$ is a right angle. In parts (i) to (iv) find the length of the remaining sides of ABC.

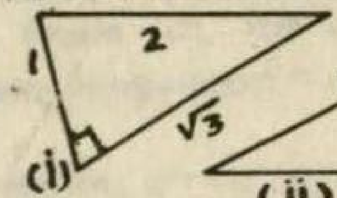
(i) $m\overline{AC} = 6$cm., $m\overline{BC} = 8$ cm.

(ii) $m\overline{BC} = 24$ units , $m\overline{AB} = 25$ units

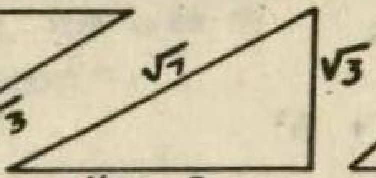
(iii) $m\overline{AC} = 9$ mm., $m\overline{AB} = 41$ mm

(iv) $m\overline{AB} = 10$ cm., $m\overline{BC} = 9$ cm.

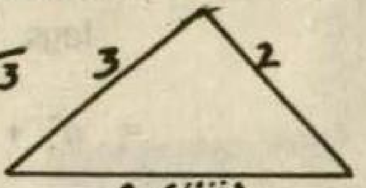
2. Which of the following triangles are right-angled ? Answer without measuring the angles.



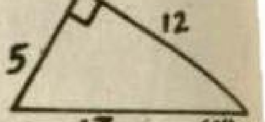
(i)



(ii)



(iii)



(v)

3. Elements of which of the following sets can represent the lengths of the sides of right-angled triangles ?

$A = \{5, 6, 7\}$, $B = \{2.5, 6, 6.5\}$

$C = \{1, 10, 3\sqrt{11}\}$ $D = \{1, 1, \sqrt{2}\}$

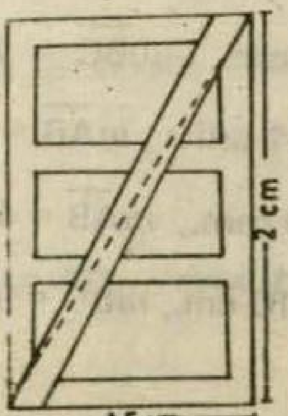
4. Compute lengths of the diagonals of the rectangular figures described below:

(i) Rectangular lawn whose length is 7.5 m. and breadth is 4 m.

(ii) A hall whose length and breadth are 20 m. and 15 m. respectively.

(iii) A play-ground whose length and breadth are 120 m. and 50 m. respectively.

5. The gate of a garden 2 m. high and 1.5 m . broad, is made up of horizontal wooden bars. To strengthen it further another bar has to be fixed along a diagonal. What should be the length of this bar?



6. A mosque is 36 m. due East of Hamid's house and a school is 48 m. due North of the house. What is the distance between the mosque and the school ?

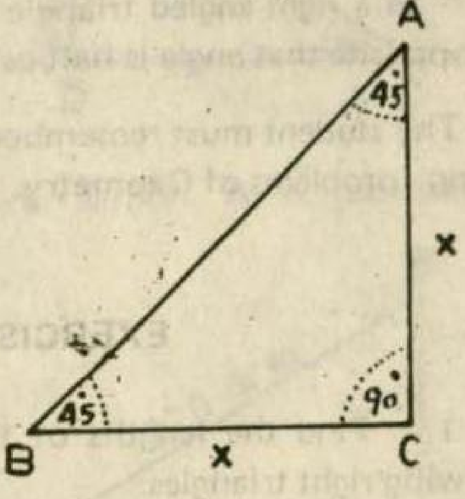
7. A labourer placed a ladder, 51 dm long, against a wall in such a way that its lower end was at a distance of 24 dm from the wall. At what height does the other end of the ladder touch the wall ?

TWO SPECIAL RIGHT-ANGLED TRIANGLES

Two special righ-angled triangles which are used by draftsmen for measuring purposes are : (i) a right-angled triangle with each of the acute angles of 45°, (ii) a right-angled triangle with a 30° acute angle.

1. In the right-angled triangle ABC, ∠B is of 45°. Is the third angle of the triangle also of 45° ?

What type of triangle is $\triangle$ ABC ? If $m\overline{AC} = x$, what is the length of $\overline{BC}$? Can we now find the length of $\overline{AB}$?



By the Pythagoras Theorem

(length of hypotenuse)^2
= (length of base)^2
+ (length of altitude)^2

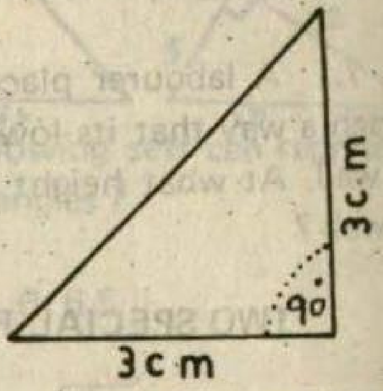
$$\therefore \quad m\overline{AB}^2 = x^2 + x^2 = 2x^2$$

$$\therefore \quad mAB = \sqrt{2x^2} = \sqrt{2}\ x$$

This shows that if the length of one leg of a 45° right triangle is known, the lengths of the remaining two sides can be determined.

Example

The length of the base of an isosceles right triangle is 3cm. What is the length of its hypotenuse ?



Solution

In the right triangle.

(length of hypotenuse)^2

= (length of base)^2

+ (length of altitude)^2

$= 3^2 + 3^2$

$= 18$

$\therefore$ length of hypotenuse $= \sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2}$ cm.

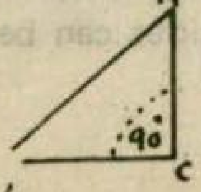
2. The right-angled triangle in which one of the acute angles is of 30° has already been discussed in the previous chapter wherein it was proved that :-

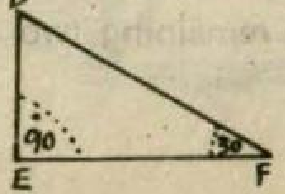
" In a right-angled triangle with one acute angle of 30°, the side opposite that angle is half as long as the hypotenuse."

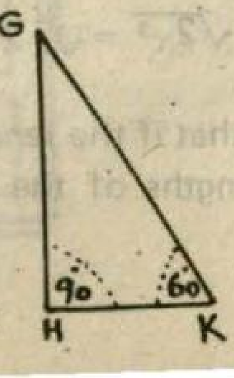
The student must remember it to be able to apply it in solving problems of Geometry.

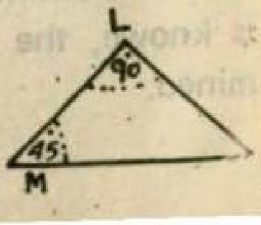
EXERCISE NO. 11.3

1. Find the lengths of the remaining sides in each of the following right triangles:









2. Does a diagonal of a square divide it into two isosceles triangles ?

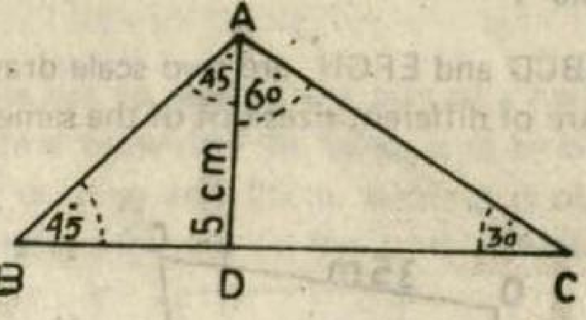
3. If the length of a diagonal of a square is $\sqrt{2}$ units, what is the length of each side ?

4. If a perpendicular is drawn from the vertex of an equilateral triangle to its base, shall we get two triangles of angles 30° 60°, 90° ?

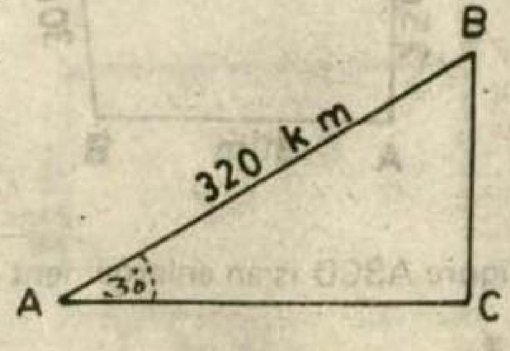
5. What is the ratio between the lengths of the shortest and the longest sides of a triangle with angles of 30°, 60°, 90° ?

6. For the sake of strength, two opposite corners of a square frame have been joined by a stretched wire. The length of the wire is 12 cm. What is the length of each side of the frame ?

7. Find the lengths of $\overline{AC}$, $\overline{AB}$, $\overline{BD}$, and $\overline{DC}$ with reference to the following figure.



8. An aeroplane flew from an airport A in the direction 30° north east and reached an airport B, 320 km. away from A. What is the distance of B from A in the east and north directions ?



9. Each side of an equilateral triangle is 8 cm. long, What is

its height ? Find the area of the region bounded by it.

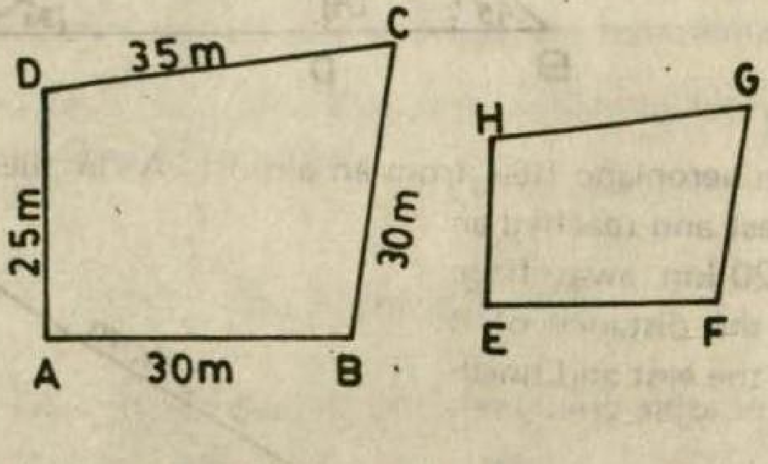
10. Each of the congruent sides of an isosceles triangular region is 16 cm. long and the measure of each of the congruent angle is 30°. Find the area of the region.

SIMILARITY OF FIGURES

You have become familiar with congruent figures, especially congruent triangles, i.e., triangles which are such that at least one (1 – 1) correspondence between them can be established in which the corresponding sides and angles of the triangles are congruent. You found that congruent triangles are alike in all respects. We shall now consider figures which are of the same shape, but not necessarily of the same size, for instance, when a photographer prepares prints of a picture which are of different sizes, all of them will be of the same shape, although of different sizes. Such figures (which are of the same shape) are called SIMILAR FIGURES.

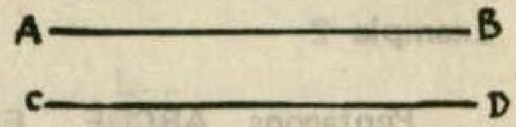
Example 1

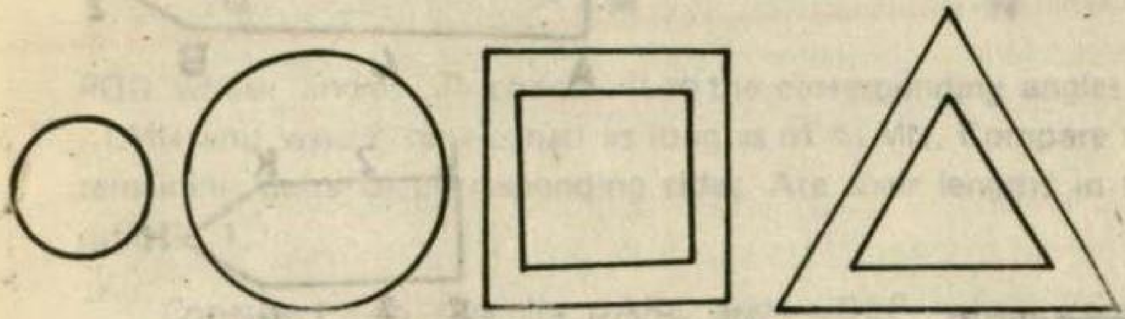
ABCD and EFGH are two scale drawings of a piece of land. They are of different sizes but of the same shape. We may say that



the figure ABCD is an enlargement of the figure EFGH.

Two figures are said to be similar when they are of the same shape. They may, or may not, be of the same size. For example, two line-segments are always similar, two equi-alteral triangles are similar, two squares are similar, two circles are similar and two congruent triangles are similar.

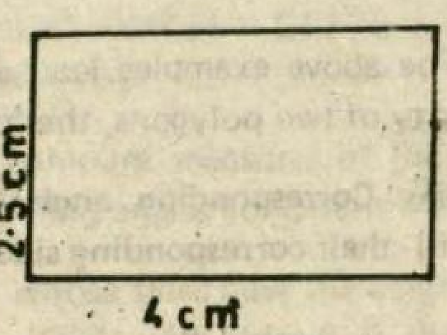




CONDITIONS OF SIMILARITY OF TWO POLYGONS

Example 1

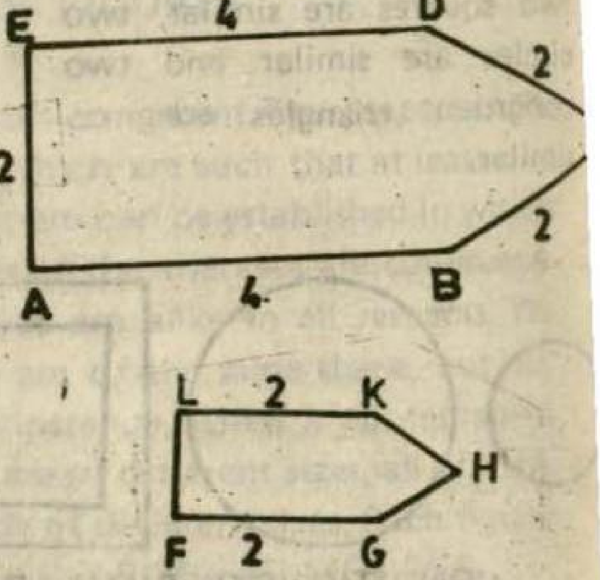
Suppose a draftsman has to prepare the plan of a house on paper and the piece of land on which the house is to be built is a rectangular region, 40 m. long and 25 m. wide. It is obvious that the plan will not be congruent to the land, May be, he represents 20 meters by 1 cm. In this way the scale drawing will be similar to the actual piece of land. Again, if he represents 10 cm. by 1 cm., the scale drawing will be similar in shape to the first one, but it will be different in size. If we compare the two plans, we find that their corresponding angles are congruent and the sides of the second plan are in length, double the corresponding



sides of the first plan, i.e., corresponding sides of the figures are proportional.

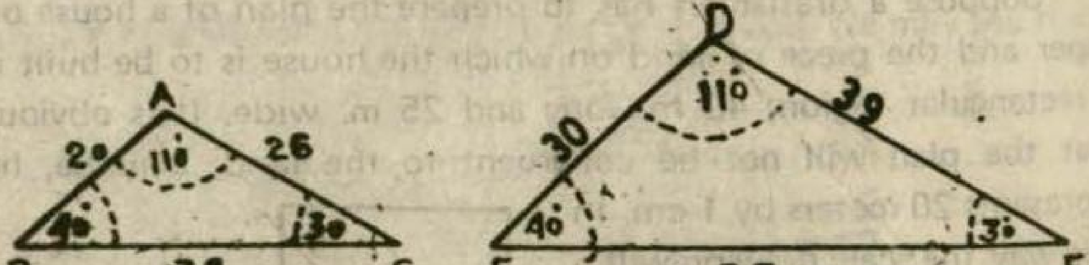
Example 2

Pentagons ABCDE, FGHKL are similar figures. Sides of one figure are double, in length, of the corresponding sides of the other figure and the corresponding angles are congruent.



Example 3

△ABC and △DEF are similar. Their corresponding angles are congruent and corresponding sides are proportional.

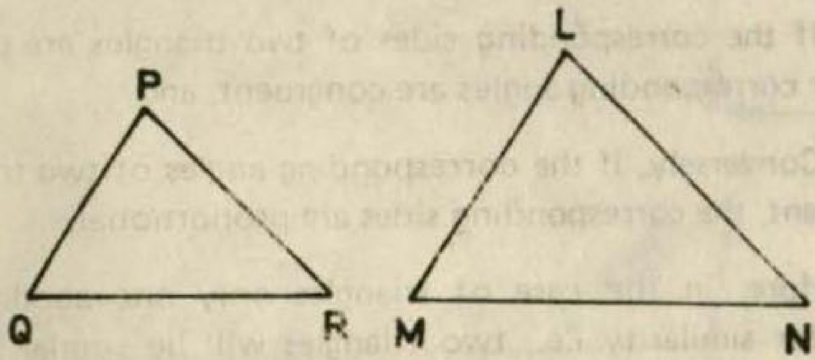


The above examples lead us to the conclusion that for the similarity of two polygons, the following conditions are necessary:

(i) Corresponding angles of the polygons are congruent; and (ii) their corresponding sides are proportional.

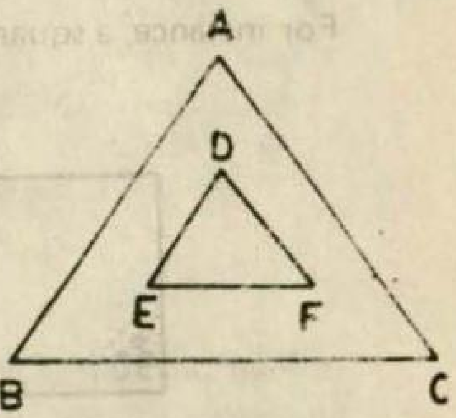
CONDITIONS FOR SIMILARITY OF TWO TRIANGLES

Construct any triangle LMN. Construct another triangle



PQR whose angles are congruent to the corresponding angles of △LMN and whose base is half as long as of △LMN. Compare the remaining pairs of corresponding sides. Are their lengths in the ratio 2 : 1

Construct two triangles △ABC and △DEF whose corresponding angles are congruent.



Compare lengths of the corresponding sides. Are they proportional ?

Do you notice that if the corresponding angles of two triangles are congruent, they are similar ?

Next construct a triangle ABC whose sides have the lengths 3, 4, 5 cm. respectively and a △ DEF whose corresponding sides have the lengths 6, 8, 10 cm., i.e., sides of △ DEF are twice as long as the corresponding sides of △ABC.

Measure their angles and compare measures of the corresponding angles. Are the corresponding angles congruent ?

Construct a triangle PQR whose sides have the lengths 2, 3, 3 cm, construct another triangle XYZ with sides 6, 9, 9 cm. Do you notice that the corresponding sides of the triangles are propor tional ? Measure the angles and compare measures of the corresponding angles. Are the corresponding angles congruent ?

The above examples give us the following conclusions:

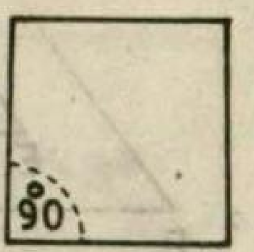
(i) If the corresponding sides of two triangles are proportional their corresponding angles are congruent, and

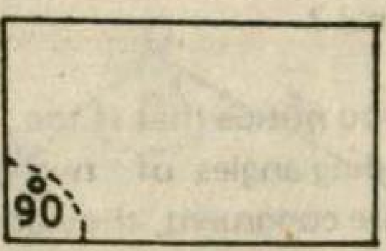
(ii) Conversely, if the corresponding angles of two triangles are congruent, the corresponding sides are proportional.

Therefore, in the case of triangles only one condition is sufficient for similarity i.e., two triangles will be similar if their corresponding angles are congruent or if their corresponding sides are proportional, because either condition implies the other.

It should be remembered that one condition of similarity is sufficient in case of triangles only and in case of other polygons both the conditions are necessary.

For instance, a square and a rectangle have their corresponding angles congruent but their sides are not proportional. Therefore, the square is not similar to the rectangle.



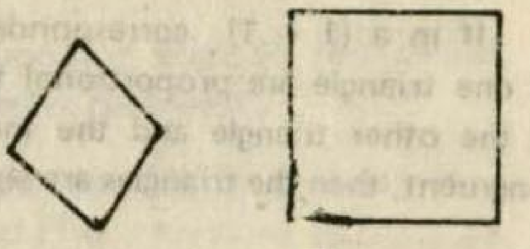


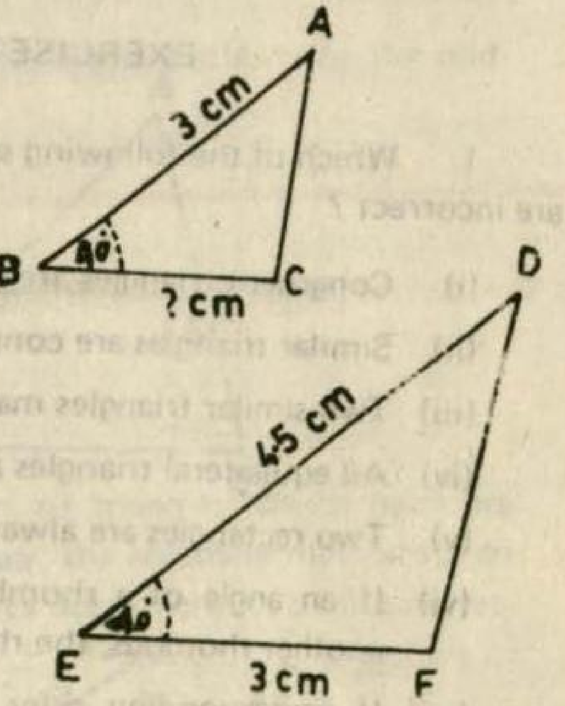
Similarly, the corresponding sides of a square and a rhombus may be proportional but they will not be similar figures (why) ?

THIRD CASE OF THE SIMILARITY OF TWO TRIANGLES

Example 1

In triangle ABC, $m\overline{AB} = 3$ $m\overline{BC} = 2$ and $m\angle B = 40°$

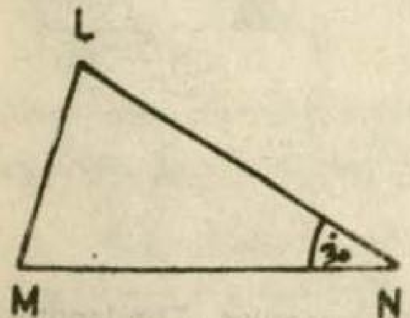


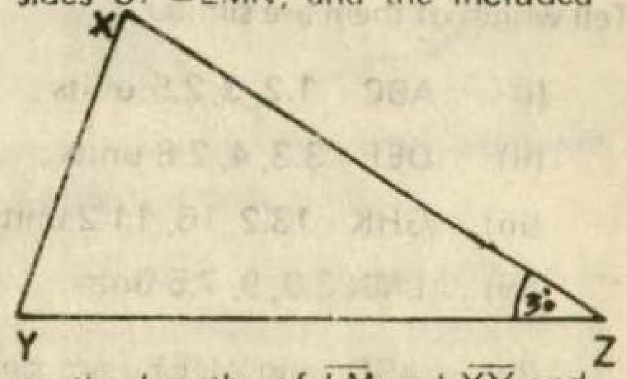


Construct a triangle DEF in which the sides $\overline{DE}$, $\overline{EF}$ are 1½ times the corresponding sides of the △ABC and measure of the included angle E is 40°. Measure the third sides of the triangles and compare their lengths. Do you notice that the ratio of $m\overline{AC}$ and $m\overline{DF}$ is the same as that of the remaining pairs of the corresponding sides. Do you draw the conclusion that the triangles are similar ?

Example 2

In ∠LMN measures of $\overline{LM}$ and $\overline{MN}$ are 4 units each and their included angle is of 30°. In △XYZ, $\overline{XY}$, $\overline{YZ}$ are 1½ times the lengths of the corresponding sides of △LMN, and the included angle is congruent to ∠N. Measure the lengths of $\overline{LM}$ and $\overline{XY}$ and verify that the ratio of their lengths is also the same as that of the other corresponding sides. Are the two triangles similar ?





Hence we have the result :

If in a (1 – 1) correspondence of two triangles, two sides of one triangle are proportional to the corresponding two sides of the other triangle and the included angles of the sides are congruent, then the triangles are similar.

EXERCISE NO. 11.4

1. Which of the following statements are correct and which are incorrect ?

(i) Congruent triangles are similar.

(ii) Similar triangles are congruent.

(iii) Two similar triangles may be congruent.

(iv) All equilateral triangles are similar.

(v) Two rectangles are always similar.

(vi) If an angle of a rhombus is congruent to an angle of another rhombus, the rhombuses are similar.

(vii) If corresponding sides of two quadrilaterals are proportional, their corresponding angles are congruent.

(viii) If corresponding sides of two triangles are proportional, their corresponding angles are congruent.

2. Below are given the lengths of the sides of four triangles. Tell which of them are similar ?

(i) $\triangle$ABC : 1.2, 3, 2.5 units ;

(ii) $\triangle$DEF : 3.3, 4, 2.8 units ;

(iii) $\triangle$GHK : 13.2, 16, 11.2 units ;

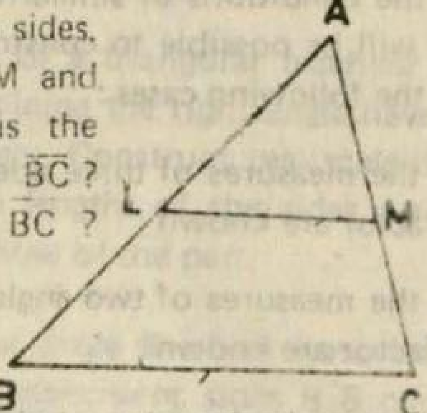
(iv) $\triangle$LMN 3.6, 9, 7.5 units.

3. $\triangle$ABC and $\triangle$DEF are similar triangles. The lengths of the sides of the triangle ABC are 6, 9, 15 cm., and the shortest side of $\triangle$DEF is 3 cm. long. Find the lengths of the remaining sides of $\triangle$DEF.

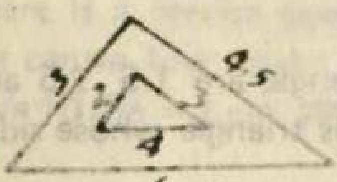
4. One pair of the corresponding sides of two similar triangles are congruent. Are the triangles congruent ?

5. The diagonals of a trapezium ABCD intersect in a point P. Prove that triangles PAB and PCD are similar.

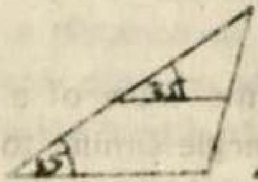
6. In a triangle ABC the line-segment LM joins the mid points L, M of two sides. Are the triangles ALM and ABC similar ? What is the ratio between $\overline{LM}$ and $\overline{BC}$? Is $\overline{LM}$ parallel to $\overline{BC}$? Justify your answers.



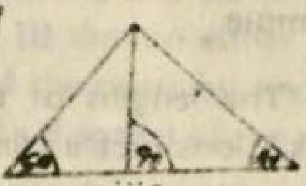
7. Below are given six pairs of triangles. Which pairs are similar ? In case of similarity of a pair, the argument may be stated like this : The corresponding angles are congruent or the corresponding sides are proportional or two corresponding sides are proportional and their included angles are congruent.



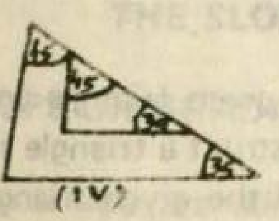
(i)



(ii)



(iii)



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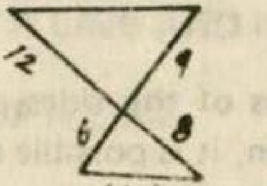

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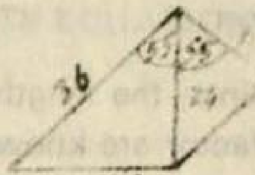


(vi)

8. The shadow cast by a tree on horizontal ground is 104 dm. long. At the same time a man notes that his shadow is 16 dm. long whereas his height is 12 dm. Using the property of similar triangles find the height of the tree.

9. How will you find the height of an electric pole without actually measuring it ?

CONSTRUCTING SIMILAR TRIANGLES

Suppose you have to construct a triangle, similar to a given triangle, i.e., you have to draw a scale drawing of the given triangle. Is it necessary that you should know the measures of all the sides and the angles of the given triangle ?

In view of the conditions of similarity of two triangles you may say that it will be possible to construct a triangle similar to given triangle in the following cases :

(1) When the measures of three sides of the triangle and the scale factor are known ;

(2) When the measures of two angles of the triangle and the scale factor are known ;

(3) When the measures of two sides of the triangle, measure of their included angle and the scale factor are known.

Example

The lengths of the sides of a triangle are 1.5, 2.5 and 3 units. Construct a triangle similar to this triangle whose sides are twice as long as those of this triangle.

Solution

Since the lengths of the sides of the given triangle and the scale factor are known, it is possible to construct a triangle similar to this triangle. The lengths of the sides of the given triangle are 1.5, 2.5, 3 units.

Therefore, the lengths of the sides of the required triangle are 3, 5, 6 units.

Hence a triangle constructed with sides 3, 5, 6 units long, is the required triangle.

EXERCISE NO. 11.5

1. The sides of a triangular piece of land are 80, 150 and 160 meters long. What are the measures of the angles at its corners ?

[Hint . Draw a scale drawing on a convenient scale and measure the angles at the vertices].

2. One angle of a triangular machine part is a right angle and the sides determining the right angle have the lengths 3 mm. and 4mm. respectively. Construct two scale drawings of the part in one of which the lengths of the sides are 5 times and in the other 10 times those of the part.

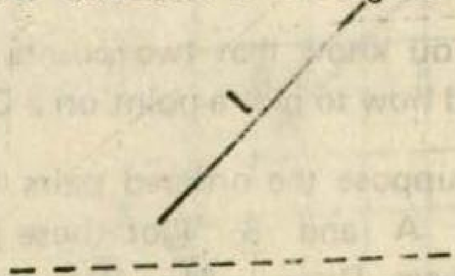
3. The vertical angle B of an isosceles triangle ABC is of 45° and each of the congruent sides is 6 cm. long. Construct a triangle PQR whose base is 2/3 of the base of $\triangle$ ABC.

4. A badminton court is 132 dm., long and 60 dm. wide. There is a service line at a distance of 18 dm. on either side of the centre line. Make a scale drawing of the court by taking the scale 10 dm = 1 cm. What is the length of a diagonal of the court?

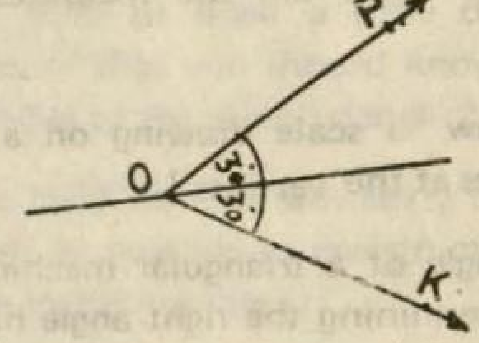
THE SLOPE OF A LINE AND ITS EQUATION

SLOPE OR GRADIENT OF A LINE

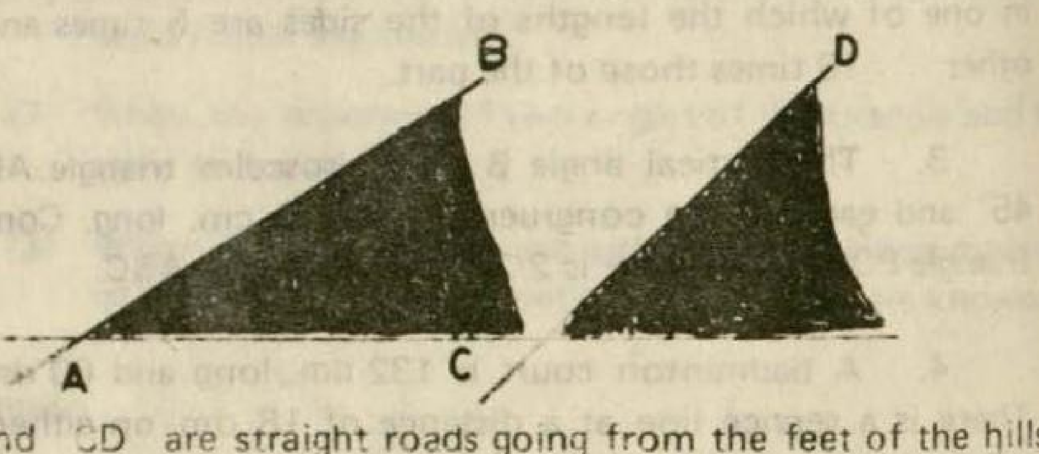
If you are asked to tell the direction a line-segment *l*, you will probably describe it in terms of the angle which it makes with a horizontal axis (x-axis,). But this method of determining direction may not prove satisfactory.



For instance, the directions of $\overrightarrow{OP}$ and $\overrightarrow{OK}$ are different although they are making angles of the same measure with the x-axis.



Consider another example of two hills and suppose that



AB and CD are straight roads going from the feet of the hills to their tops.

We want to know as to which hill is easier to climb up. In other words, we would like to know the steepness or inclination of AB and CD. Mathematically we say that we need to know the slope or Gradient of the line segments.

In this section we will discuss the method of finding the slope of a line geometrically.

You know that two points determine a line. You have also studied how to plot a point on a Cartesian plane.

Suppose the ordered pairs (x_1, y_1), (x_2, y_2) represent two points A and B. Plot these points on the Cartesian plane and join them. Draw $\overline{AL}$, $\overline{BM}$ perpendicular to x-axis, and drop $\overline{AP}$ perpendicular from A to $\overline{BM}$.

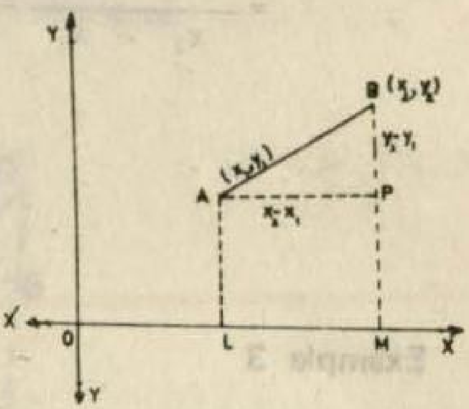
The ratio $\frac{PB}{AP}$ will be said to be the slope of $\overleftrightarrow{AB}$ (or $\overline{AB}$).

Now $m\overline{PB} = y_2 - y_1$ and

$m\overline{AP} = x_2 - x_1$

$$\frac{m\overline{PB}}{m\overline{AP}} = \frac{y_2 - y_1}{x_2 - x_1}$$

(provided $x_2 - x_1 \neq 0$)



Thus the slope of AB

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

$\frac{y_2 - y_1}{x_2 - x_1}$ is a real number

$\therefore$ It can assume positive, zero or negative value. Therefore, the slope of a line (line segment) may be positive, negative or zero.

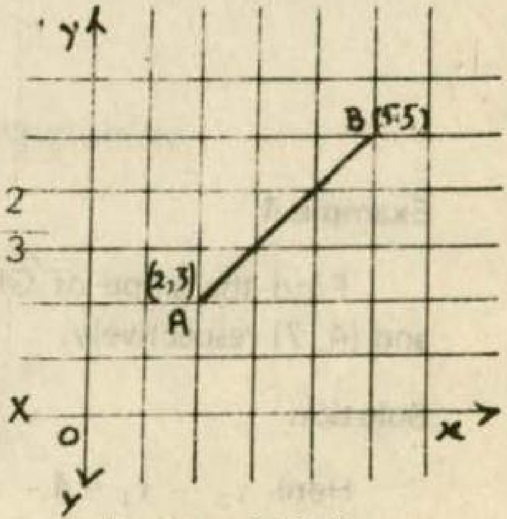
Example 1

The co-ordinates of points A and B are (2, 3) and (5, 5) respectively. Find the slope of $\overline{AB}$.

Solution

Slope of $\overline{AB}$

$$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 3}{5 - 2} = \frac{2}{3}$$



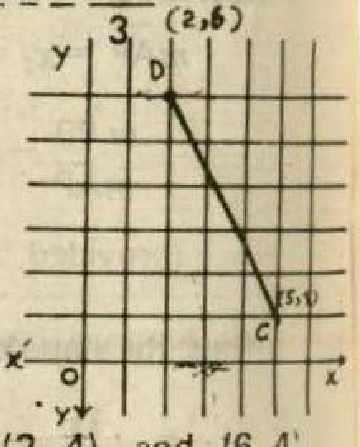
Example 2

Find the slope of $\overline{CD}$ where the co-ordinates of C, D are respectively (5, 1) and (2, 6).

Solution

Slope of $\overline{CD}$

$$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-1}{2-5} = \frac{5}{-3} = -\frac{5}{3}$$



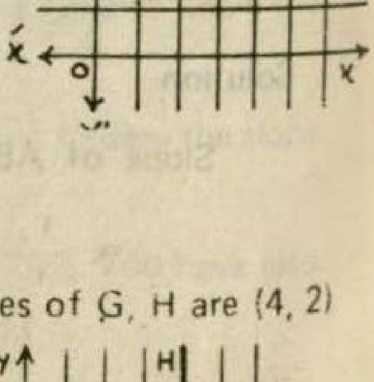
Example 3

Co-ordinates of points E and F are (2, 4) and (6, 4) respectively. What is the slope of $\overline{EF}$?

Solution

Slope of $\overline{EF}$

$$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-4}{6-2} = 0$$



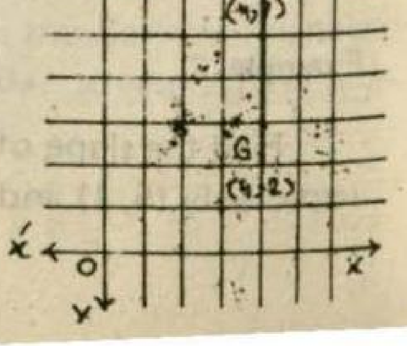
Example 4

Find the slope of $\overline{GH}$ when co-ordinates of G, H are (4, 2) and (4, 7) respectively.

Solution

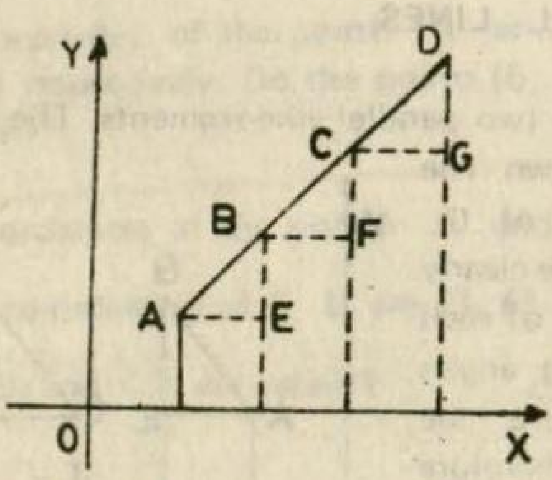
Here $x_2 - x_1 = 4 - 4 = 0$

Therefore, slope of GH cannot be determined.



SLOPE OF COLLINEAR LINE-SEGMENTS

$\overline{AB}$, $\overline{BC}$ and $\overline{CD}$ are collinear line-segments.



The corresponding angles of the triangles AEB and BFC are congruent. Therefore, these are similar triangles. So their corresponding sides are proportional.

$$\therefore \frac{m\overline{BE}}{m\overline{CF}} = \frac{m\overline{AE}}{m\overline{BF}}$$

By multiplying each side of this equation by

$\dfrac{m\overline{CF}}{m\overline{AE}}$, we get

$$\frac{m\,\overline{BE}}{m\,\overline{AE}} = \frac{m\overline{CF}}{m\,\overline{BF}}$$

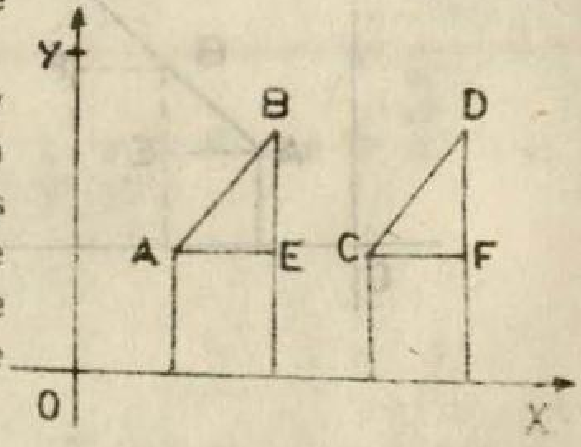
i.e. the slope of $\overline{AB}$ is equal to the slope of $\overline{BC}$. Similarly, triangle CGD is similar to each of the triangles AEB and BFC. Therefore, the slope of $\overline{CD}$ is equal to the slope of $\overline{BC}$ and

$\overline{AB}$. Thus we get the result :

Collinear line-segments have the same slope.

SLOPE OF PARALLEL LINES

Let $\overline{AB}$, $\overline{CD}$ be two parallel line-segments. The figure has been completed as shown. The corresponding angles of the triangles AEB, CFD are clearly congruent. (The arms of each pair of corresponding angles are parallel). Therefore, the triangles are similar. Therefore the corresponding sides are proportional and so



$$\frac{m\overline{BE}}{m\overline{DF}} = \frac{m\overline{AE}}{m\overline{CF}}$$

Multiplying each side of this equation by

$$\frac{m\overline{DF}}{m\overline{AE}}, \text{ we get,}$$

$$\frac{m\overline{BE}}{m\overline{AE}} = \frac{m\overline{DF}}{m\overline{CF}}$$

i.e. the slope of $\overline{AB}$ is equal to the slope of $\overline{CD}$. Hence the slope of parallel lines is also the same. The criterion for parallelism of two lines is that their slope is the same.

EXERCISE NO. 11.6

1. Plot the following pairs of the points on the Cartesian

plane and find the slopes of the lines determined by them:

(i) (2, 2), (5, 8)   (ii) (−4, 1), (5, −2)

(iii) (10, 3), (−3, 3);   (iv) (3, 0), (0, −2)

2. AB is parallel to the x-axis. What is its slope ?

3. Co-ordinates of the points A and B are (−7, 3) and (−5, −2) respectively. Do the points (6, 10) and (−9, 8) lie on the line AB ?

4. Co-ordinates of the points A and B are (2, 3) and (8, 5) and co-ordiantes of C, D are (3, 6), (6, 7) respectively Show that $\overleftrightarrow{AB}$. and $\overleftrightarrow{CD}$ are parallel.

EQUATION OF A LINE

You have studied in Algebra that a linear equation in two variables represents a line. Therefore, all the points which lie on the line represented by the equation constitute the solution set of the equation.

Now we shall find the method of writing the equation of a line when co-ordinates of one of its points and its slope are known.

Example

Write the line AB in the equation form when $\overleftrightarrow{AB}$ passes through the point (1, 2) and its slope is $\frac{2}{3}$.

Solution

Let (x, y) be any point on $\overleftrightarrow{AB}$

Slope of $\overleftrightarrow{AB} = \dfrac{y-2}{x-1}$

$$\therefore \quad \frac{y-2}{x-1} = \frac{2}{3}$$

or $\quad y - 2 = \dfrac{2}{3}(x-1) \qquad (1)$

or $\quad 3y - 6 = 2x - 2$

or $\quad 2x - 3y + 4 = 0$

which is the required equation.

From equation (1) it is follows that the equation of the line through the point (x, y) with slope m is

$$y - y_1 = m(x - x_1)$$

EXERCISE NO.11.7

1. Line AB passes through the point (1, 2) and its slope is $-\dfrac{3}{2}$. Write it in the equation form.

2. Write down the equation of the line which passes through the point (1, 3) and whose slope is 1.

3. Line LM passes through (0, 0) and its slope is 0. Find its equation.

4. Find the slope of the line and the co-ordinates of a point on the line represented by each of the equations:

(i) $y - 3 = \dfrac{2}{5}(x - 0)$; (ii) $y + 2 = \dfrac{5}{2}(x - 3)$

(iii) $y = x$; (iv) $y = 3x$;

(v) $y + 6 = x$

CIRCLE

BASIC DEFINITIONS

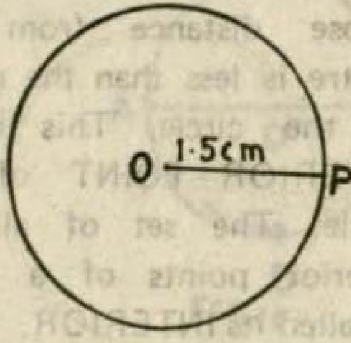
You have acquired some information about the circle in the lower classes. We state some definitions for revision and add some new definitions also.

Circle

A circle is a set of points each of which is at the same distance from a fixed point. The fixed point is called the CENTRE of the circle, and the distance of each point of the circle from the point is called the RADIUS of the circle.

In figure (1) a circle is shown. Its centre is O and its radius is 1.5 cm. If P is any point of the circle,

$$m\overline{OP} = 1.5 \text{ cm.}$$



Fig·1

Radial Segment

In figure (1) the line-segment OP is such that its one end-point is the centre of the circle and the other end point is a point of the circle. Such a line-segment is called a radial segment.

The end-point of radial segment which is on the circle is called its OUTER END-POINT. In figure (1) the outer end-point of $\overline{OP}$ is the point P.

Chord

A line-segment whose end-points are two points of a circle is called a CHORD in the circle. In figure (2) $\overline{PQ}$ is a chord.

Diameter

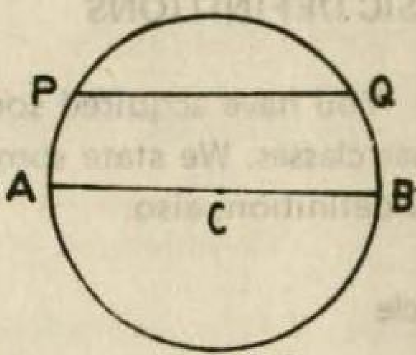


Fig.2

In figure (2) AB is a special chord. It contains the centre of the circle. Such a chord is called a DIAMETER of the circle. A diameter of a circle is a chord which contains (passes through its centre.

Evidently the length of a diameter is twice that of a radial segment.

Interior of a Circle

In figure (3) is shown a circle with centre O. A is a point whose distance from the centre is less than the radius of the circle. This is an INTERIOR POINT of the circle. The set of all the interior points of a circle is called its INTERIOR.

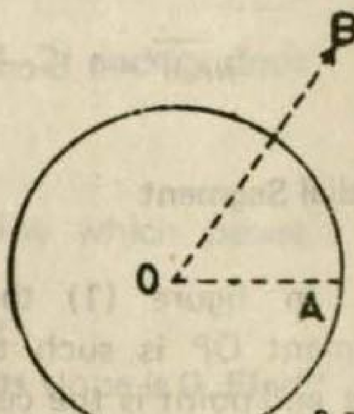
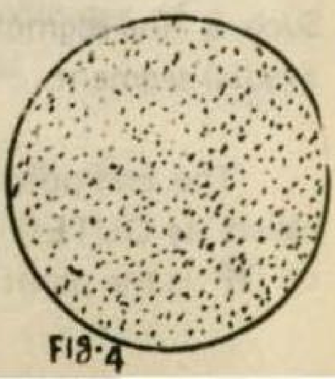


Fig.3

In figure (4) the interior of the circle has been shaded.

Remember that the points on a circle are not contained in its interior.

In figure (3) B is a point whose distance from the centre of the circle is more than its radius. It is an EXTERIOR POINT.



Fig·4

Arc of a Circle

In figure (5), A, B are two points of the circle. These points divide the circle into two parts. One part is a shorter path from A to B (along the circle) whereas the other part is a longer path. The shorter path is called a MINOR ARC and the longer path is called a MAJOR ARC. The minor arc is denoted as $\overset{\frown}{AB}$ and the major arc is denoted as $\overset{\frown}{ACB}$ where C is a point on the circle not belonging to the minor arc. The arcs may also be written $\overset{\frown}{BA}$, $\overset{\frown}{BCA}$. The points A, B are supposed to be contained in each of the arcs and are called their END-POINTS i.e., the points A, B are the end-points of $\overset{\frown}{AB}$ as well as $\overset{\frown}{ACB}$.

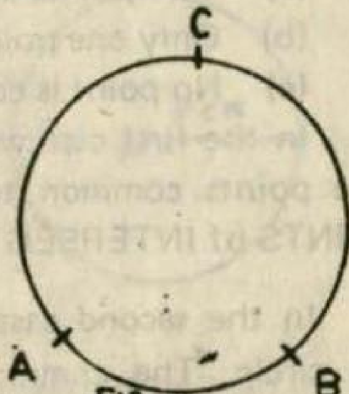


Fig. 5

Semi Circle

In figure (6) A, B are the end-points of a diameter, $\overline{AB}$. These points divide the circle into two parts which are congruent. Each of such parts is called a SEMI-CIRCULAR ARC or simply a SEMI-CIRCLE. In the figure $\overset{\frown}{ACB}$ and $\overset{\frown}{ADB}$ are semi-circles. The points A, B are contained in both the semi-circles and are called END-POINTS of each of them.

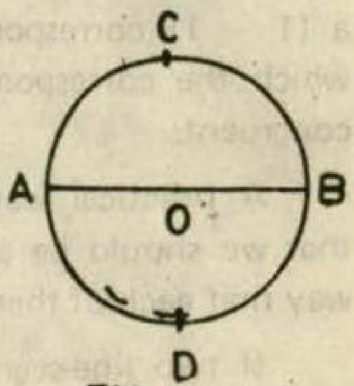


Fig. 6

Secant and Tangent

As for the intersection of a circle and a line l, any one of the

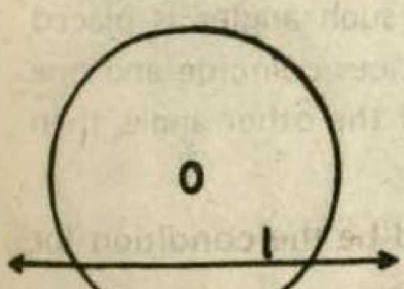


Fig. 7a.

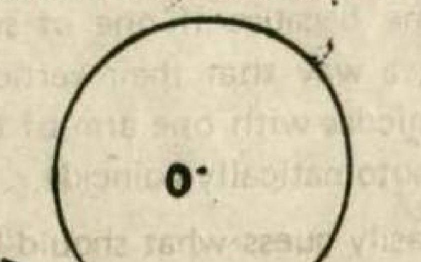


Fig. 7b.

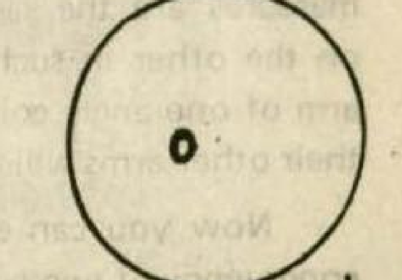


Fig. 7c.

following three situations can arise :

(a) Two points are common to *l* and the circle.

(b) Only one point is common to *l* and the circle.

(c) No point is common to them.

In the first case we may say that *l* is a SECANT of the circle. The points common to the line and the circle are called their POINTS of INTERSECTION.

In the second case we say that the line *l* is a TANGENT of the circle. The common point T is called the POINT OF CONTACT of the line and the circle. We also say that the line *l* TOUCHES the circle at the point T.

In the third case we say that the line and circle are NON-INTERSECTING. (We may say that their intersection is an empty set).

Congruent Circles

You know that two triangles are said to be congruent if a (1 – 1) correspondence can be established between them in which the corresponding sides and the corresponding angles are congruent.

A practical way of testing congruence of two triangles is that we should be able to put one figure on the other in such a way that each of them coincides with the other.

If two line-segments are of equal length they can be made to coincide with each other. Therefore, two line-segments are said to be congruent if they are of the same length.

Similarly, two angles are said to be congruent if their degree measures are the same because if one of such angles is placed on the other in such a way that their vertices coincide and one arm of one angle coinicdes with one arm of the other angle, then their other arms will automatically coincide.

Now you can easily guess what should be the condition for congruence of two circles.

In figure (8) are shown two circles with centres O, O′ and the radius of each is 4 cm.

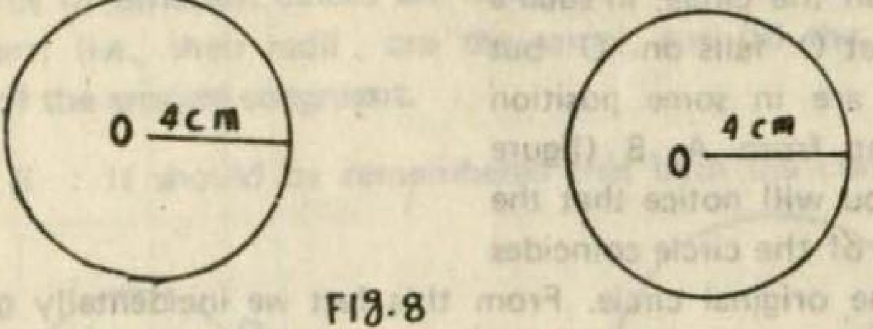


FIG.8

If we trace the circle with centre O and place it on the circle with centre O′ such that the centres of the circles coincide, we shall find that the circles exactly coincide with each other. Therefore,

Two circles are congruent when their radii are equal.

Central Angle of one Arc

In figure (9) AB is a minor arc of the circle with centre O. The rays $\overrightarrow{OA}$, $\overrightarrow{OB}$ make the angle AOB. This is the central angle of the arc AB. In other words.

The central angle of an arc is the union of two rays that pass through the centre of the circle and the end-points of the arc.

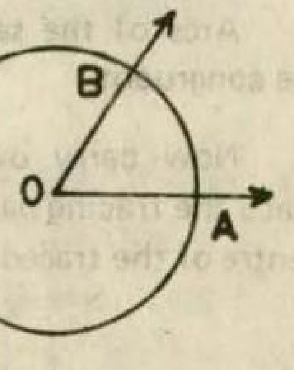
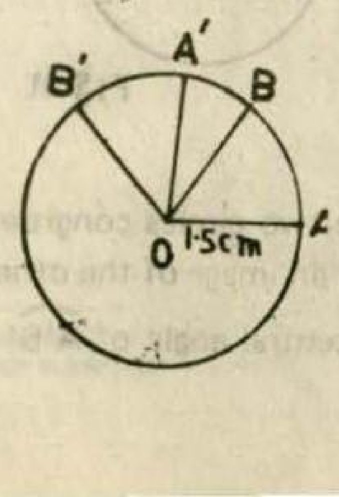


FIG. 9

Congruent Arcs

Draw a circle with centre O and radius equal to 1.5cm. Mark any two points A, B on it. Trace the circle on tracing paper. Let the pictures of O, A, B on the tracing paper be O′, A′, B′ respec-



tively. Now place the tracing paper on the circle, in such a way that O′ falls on O but A′, B′ are in some position different from A, B (figure 10). You will notice that the picture of the circle coincides with the original circle. From this fact we incidentally get the result that every circle is congruent to itself. Now look at the figure and answer the following questions;

Is the central angle of $\overset{\frown}{A'B'}$ congruent with the central angle of $\overset{\frown}{AB}$? (Yes).

Is the length of the A′B′ equal to the length of $\overset{\frown}{AB}$?

Is it possible to move the traced circle in such a way that O′ remains coincident with O and $\overset{\frown}{A'B'}$ coincides with $\overset{\frown}{AB}$?

Answers to these questions give us the result ;

Arcs of the same circle are congruent if their central angles are congruent.

Now carry out another experiment with the same circle. Place the tracing paper on the original circle in such a way that the centre of the traced circle O′ is away from O (figure 11).

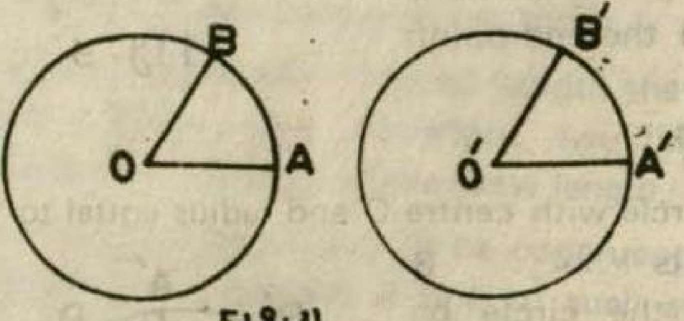


Fig. 11

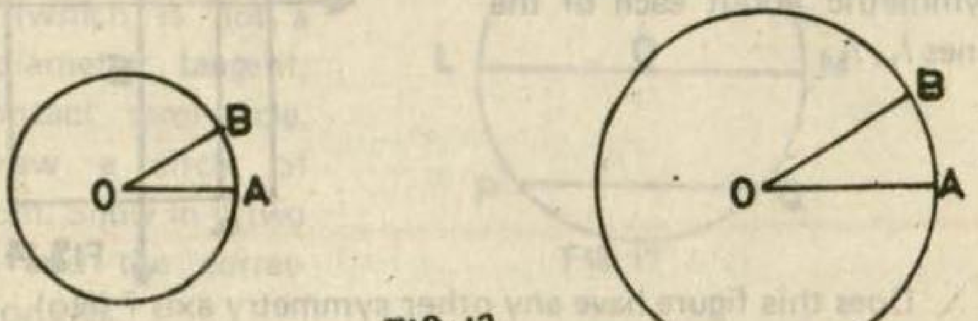
Are the two circles congruent ? (Doubtlessly so, because one circle is only an image of the other).

Is the central angle of $\overset{\frown}{A'B'}$ congruent to the central angle of $\overset{\frown}{AB}$. ?

This experiment yields the result :

Arcs of different circles are congruent if (i) the circles are congruent (i.e., their radii are the same), and (ii) the central angles of the arcs are congruent.

N.B. : It should be remembered that both the conditions stated above are necessary for congruence of arcs of non-concentric circles. If two circles are of unequal radii, their arcs, having congruent central angles, will not be congruent. The arc of the bigger circle will be greater in length than the arc of the smaller circle, and the arcs cannot be made to coincide with each other.



Fig·12

Axial Symmetry of the Circle

The adjoining figure is such that if we fold it along the line *l*, the two parts will coincide with each other. We say the figure is symmetric about the line *l*, which is called its axis of symmetry.*

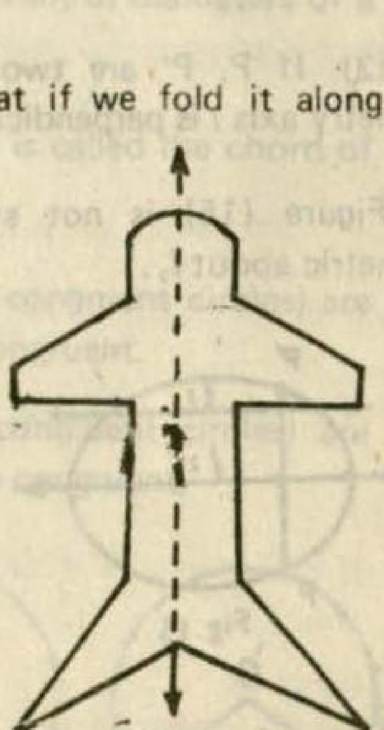


Fig-13

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* It is not necessary for the symmetry to be a whole line. A line-segment can also be a symmetry axis provided its length is not less than the length of the figure(in the intersection of the axis).

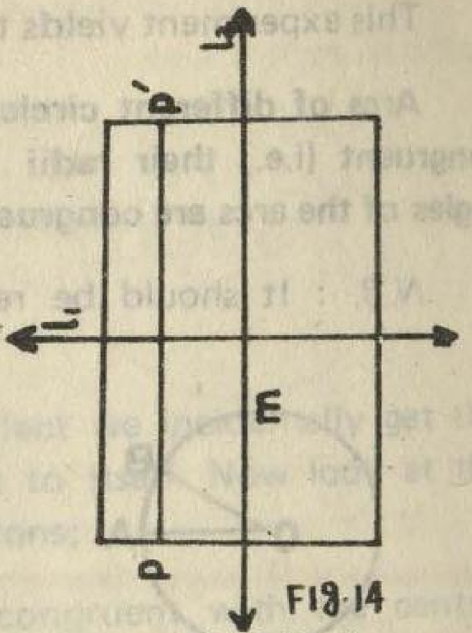


FIG. 14

A rectangle is shown in figure 14. Lines l_1, l_2 pass through the mid-points of its opposite sides. The rectangle is symmetric about each of the lines l_1, l_2.

Does this figure have any other symmetry axis ? (No).

Another criterion for axial symmetry is this : Corresponding to any point P of the figure there exists a point P′ of the figure such that l is the perpendicular bisector of PP′ (figure 14).

From the above discussion we also deduce that :

(1) If P, P′ are two points of the figure such that the symmetry axis l bisects $\overline{PP'}$, then l is perpendicular to $\overline{PP'}$.

(2) If P, P′ are two points of the figure such that the symmetry axis l is perpendicular to $\overline{PP'}$, then l also bisects $\overline{PP'}$.

Figure (15) is not symmetric about l_1 (why ?). It is symmetric about l_2.

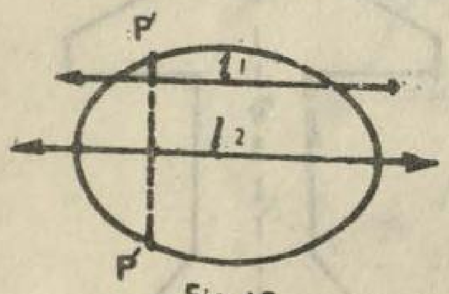


Fig. 15

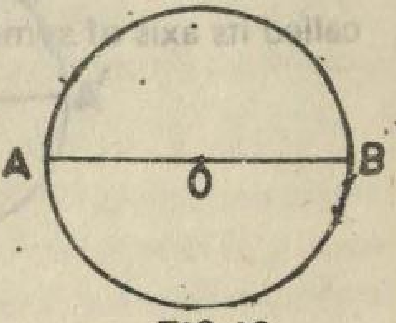


FIG. 16

A circle is shown in figure (16). Its diameter $\overline{AB}$ has also been shown. Is the circle symmetric about $\overline{AB}$?

A circle is symmetric about each of its diameters.

EXERCISE NO.11.8

1. In the adjoining figure (Fig. 17) name the following :

Chord (which is not a diameter); diameter; tangent; point of contact; semi-cricle.

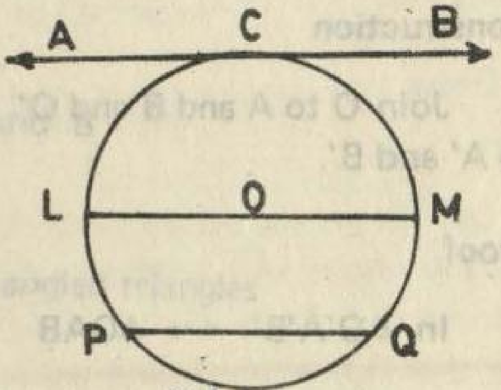


FIG. 17

2. Draw a circle of radius of 5 cm. Show in it two minor arcs and the corresponding major arcs.

3. Is a semi-circle a minor arc or a major arc ?

4. Is every diameter of a circle its chord ? Is every chord a diameter ?

5. How many diameters and how many chords can be drawn through a point of a circle ? Is the set of diameters of a circle a sub-set of the set of its chords ?

6. If $\overset{\frown}{AB}$ is an arc of a circle, AB is called the chord of the arc. Prove that :

(a) If two chords of a circle (or of congruent circles) are congruent, the corresponding arcs are also congruent.

(b) If two arcs of a circle (or of congruent circles) are congruent, the corresponding chords are also congruent.

Solution (a)
Given

In two congruent circles with centres O, O',

$\overline{AB} \cong \overline{A'B'}$

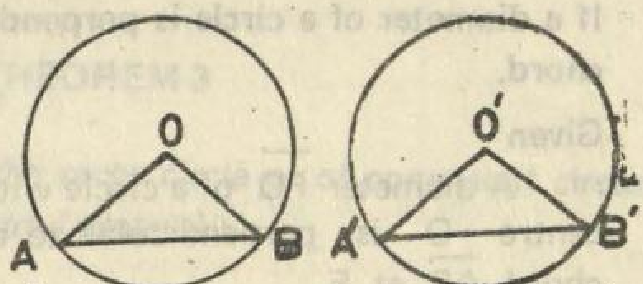


FIG. 18

To Prove

$\overset{\frown}{AB} = \overset{\frown}{A'B'}$

Construction

Join O to A and B and O′ to A′ and B′.

Proof

In $\triangle O'A'B' \longleftrightarrow \triangle OAB$

$O'A' \cong OA$...........(Radii of congruent circles)

$O'B' \cong OB$ (Radii of congruent circles)

$A'B' \cong AB$ (Given)

$\therefore \quad \triangle O'A'B' \cong \triangle OAB$

and $\angle O' \cong \angle O$

Thus $\overset{\frown}{AB}$ and $\overset{\frown}{A'B'}$ are arcs of congruent circles and their central angles are congruent.

$\overset{\frown}{A'B'} \cong \overset{\frown}{AB}$

Part (b)

Write out the proof yourself. Use the theorem

S.A.S ≅ S.A.S

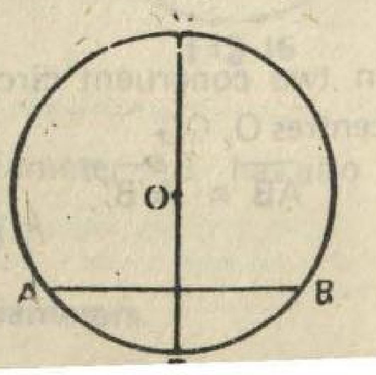
2. SOME THEOREMS ABOUT CIRCLES

THEOREM I

If a diameter of a circle is perpendicular to a chord, it bisects the chord.

Given

A diameter $\overline{PQ}$ of a circle with centre O is perpendicular to a chord $\overline{AB}$ at E.



To Prove

$\overline{PQ}$ bisects chord $\overline{AB}$ at E

or $\overline{AE} \cong \overline{EB}$

Construction : Join O to A and B

Proof:

In correspondence of right-angled triangles

$\Delta OEA \leftrightarrow \Delta OEB$

$\overline{OA} \cong \overline{OB}$ (radial segments of the circle)

$\overline{OE} \cong \overline{OE}$ (Common)

$m\angle OEA = m\angle OEB$ (90° each)

Hence $\Delta OEA \cong \Delta OEB$ (H.S. $\cong$ H. S.)

$\therefore$ $\overline{AE} \cong \overline{EB}$

*Cor.** : If a radial segment of a circle is perpendicular to a chord, it also bisects the chord.

THEOREMS 2

If a diameter of a circle bisects a chord, it is perpendicular to the chord.

Proof

Write out the proof yourself.

THEOREM 3

Congruent chords of the same circle or of congruent circles are equidistant from the centre (centres).

**Cor.* : (Abbreviation of Corrolary) is a result which follows immediately from the theorem without elaborate arguments or proof.

For explaining and proving ~~this~~ theorem we make an experiment.

Experiment

Draw on paper a circle of any radius.

Let O be the centre of circle. Draw any chord $\overline{AB}$, and drop $\overline{OL}$ perpendicular from O on $\overline{AB}$. Trace the figure on tracing paper. Place the picture of the figure on it in such a way that O′, the picture of O, coincides with O but the pictures of A, B, $\overline{OL}$ take new positions A′, B′, OL′, respectively.

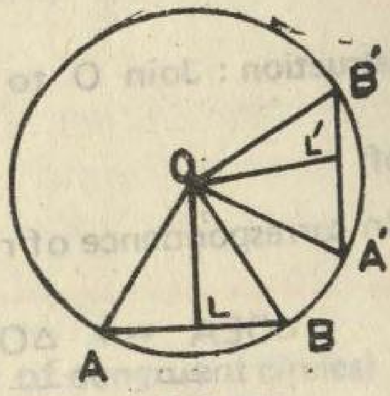


Fig. 20

Is the chord A′B′ congruent to the chord AB ? (Yes).

Is the perpendicular OL′ congruent to OL ? (Yes)

The theorem becomes quite evident by this experiment.

Alternative Proof

The theorem may also be proved as follows :

Given

In a circle with centre O, chord A′B′ is congruent to chord $\overline{AB}$, i.e., $m\overline{A'B'} = m\overline{AB}$.

To Prove

$\overline{A'B'}$ and $\overline{AB}$ are equidistant from O.

Construction

Join O to A, B and A′, B′. Draw perpendicualrs OL, $\overline{OL}$ m O to $\overline{AB}$ and $\overline{A'B'}$ respectively.

Proof

In $\triangle O'A'B' \leftrightarrow \triangle OAB$

$\overline{OA'} \cong \overline{OA}$ (Radii of the same circle)

$\overline{OB'} \cong \overline{OB}$ (Radii of the same circle)

$\overline{A'B'} \cong \overline{AB}$ (Given)

$\therefore$ $\triangle OA'B' \cong \triangle OAB$ (S.S.S. ≅ S.S.S.)

$\therefore$ $\overline{OL'} \cong \overline{OL}$ (Corresponding prependiculars of congruent triangles are congruent).

N.B. : The same proof (with slight modification) is valid for congruent chords of congruent circles. Draw the figure and write the proof yourself.

THEOREM 4

A line which is perpendicular to a radial segment of a circle at its outer end-point is a tangent of the circle.

Before giving the proof of this theorem we make an experiment.

Experiment

Draw a circle on a piece of paper or card-board. Let $\overline{AB}$ be a diameter of the circle. Take a thin and stiff piece of wire. Place it on the figure such that it is perpendicular to $\overline{AB}$ at O. Slide it slowly towards one side in such a way that its direction does not change and it remains in contact with the paper (or card-board). Notice that the wire is always perpendicular to $\overline{AB}$ and the

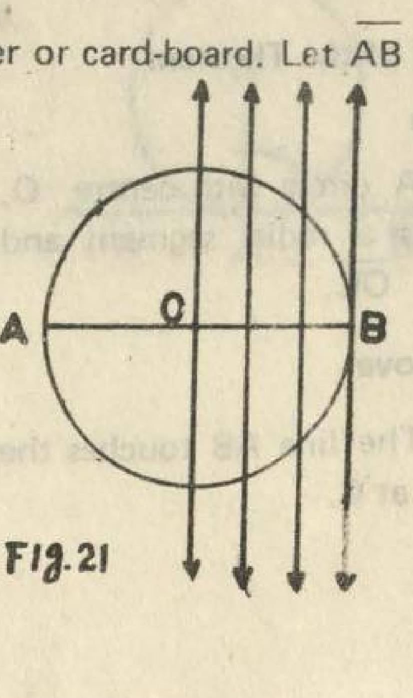


Fig.21

distance between points of intersection of the wire and the circle goes on becoming less and less until both the points of intersection coincide at B. In this position also the wire is perpendicular to $\overline{AB}$ (or $\overline{OB}$) and touches the circle at B.

From the experiment it follows that if a line is perpendicular to a radial segment at its outer end-point, it is a tangent to the circle.

In order to prove this theorem we shall require the following two results – which we accept as true without proof:

(1) If two angles of a triangle have unequal measures the side opposite to the bigger angle is longer than the side opposite to the smaller angle.

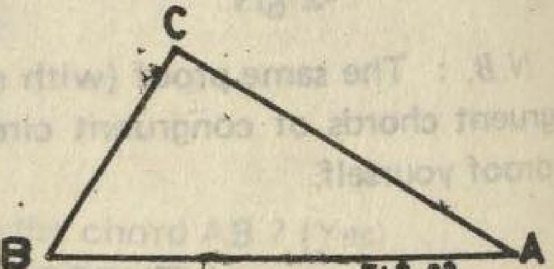


Fig.22

For example in the triangle ABC

$$m\angle B > m\angle A$$

$$\therefore\ m\overline{AC} < m\overline{BC}$$

(2) One and only one perpendicular can be drawn to a line from an outer point.

P

Proof of the Theorem

Given

A circle with centre O, $\overline{OC}$ is a radial segment and $\overleftrightarrow{AB} \perp \overline{OC}$.

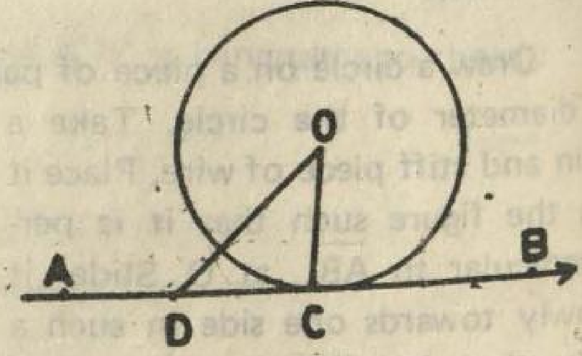


Fig.23

To prove

The line AB touches the circle at C.

Construction

Take any point D on $\overleftrightarrow{AB}$ distinct from C and join it to O.

Proof

In the triangle OCD

$m\angle C = 90^\circ$ (given)

$\therefore\ m\angle D < 90^\circ$

$\because\ m\angle C > m\angle D$

$\therefore\ m\overline{OD} > m\overline{OC}$

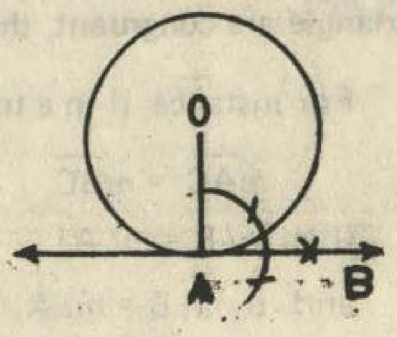
But $\overline{OC}$ is a radial segment.

Therefore, the point D is outside the circle.

Similarly, we can show that any point other than C lying on $\overleftrightarrow{AB}$ is outside the circle. Therefore, the line AB is a tangent to the circle at the point C.

CONSTUCTION THEOREM !

To draw a tangent at a point of a circle.



Given

A circle with centre O and a point A on it.

Required

To construct tangent to the circle at point A.

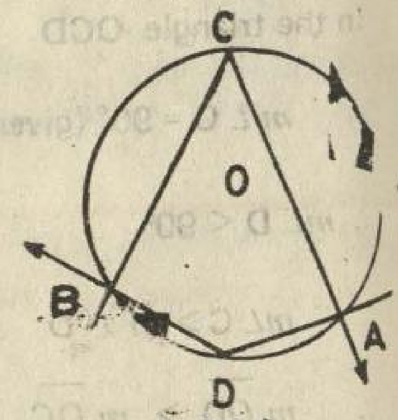
Constuction

Joint O to A and with the help of a ruler and compasses draw the perpendicular to OA at A. The perpendicular $\overleftrightarrow{AB}$ is the required tangent.

N.B. : In the light of Theorem 4, it is clear that $\overleftrightarrow{AB}$ is the tangent to the circle at A.

Before proving the next theorem we lay down a definition.

Inscribed Circle of an Arc



Fig·24

A circle with centre O is shown in the adjoining figure. $\widehat{AB}$ is a minor arc of this circle and $\widehat{ACB}$ is a major arc.

The angle ADB is such that its arms pass through the end-points of the minor arc AB and vertex is a point of the said arc. Such an angle is called an INSCRIBED ANGLE of the arc. Angle ACB is such that its arms pass through the end-points of the major arc ACB and its vertex is a point C of this arc. This is an inscribed angle of the major arc ACB.

In the course of the proof of the next theorem , we shall require the following results as well:

(a) If two sides of a triangle are congruent, then the angles opposite to them are also congruent. Conversely if two angles of a triangle are congruent, the opposite sides are also congruent.

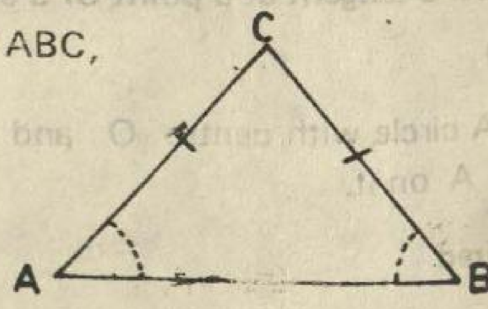
For instance, if in a triangle ABC,

$$m\overline{AC} = m\overline{BC}$$

then $m\angle B = m\angle A$

and if $m\angle B = m\angle A$,

then $m\overline{AC} = m\overline{BC}$



(b) The measure of the exterior angle determined by producing one side of a triangle is equal to the sum of the measures of the two opposite interior angles.

For example, in figure (26)

$$m\angle GEF = m\angle D + m\angle F$$

(Can you prove this result ?)

Now we are in a position to state and prove the next theorem.

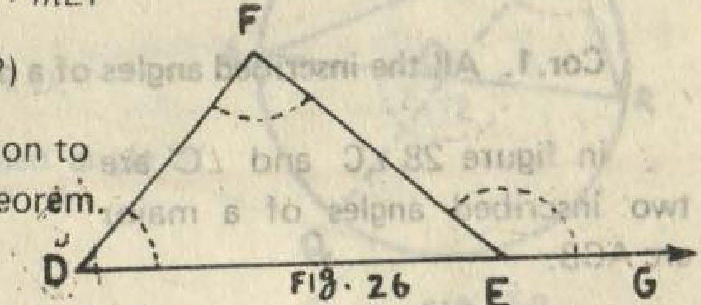


Fig. 26

THEOREM 5

The measure of a central angle of a minor arc is double the measure of the inscribed angle of the corresponding major arc.

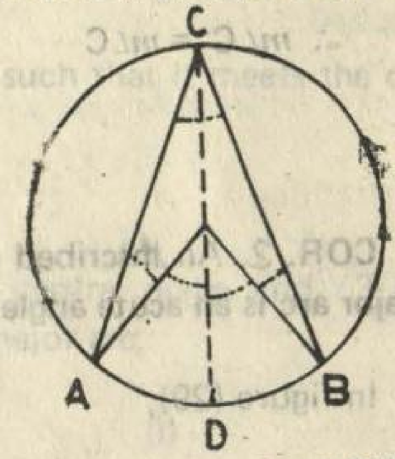


Fig.27

Given

A circle with centre O. $\angle AOB$ is the central angle of the minor arc AB.

$\angle ACB$ is an inscribed angle of the major arc ACB.

To prove

$$m\angle AOB = 2m\angle ACB$$

Consturction

Joint C to O and produce $\overrightarrow{CO}$ such that it meets the circle again in point D.

Proof

In $\triangle AOC$, $m\overline{AO} = m\overline{OC}$

$\therefore \quad m\angle 1 = m\angle 2$

But $m\angle 3 = m\angle 1 + m\angle 2$

$\therefore \quad m\angle 3 = 2m\angle 1 \qquad \ldots \qquad$ (i)

Similarly, $m\angle 6 = 2m\angle 4$ (ii)

Adding the respective sides of (i) and (ii)

$$m\angle 3 + m\angle 6 = 2(m\angle 1 + m\angle 4)$$

$$\therefore \quad m\angle AOB = 2m\angle ACB$$

Cor.1. All the inscribed angles of a major arc are congruent.

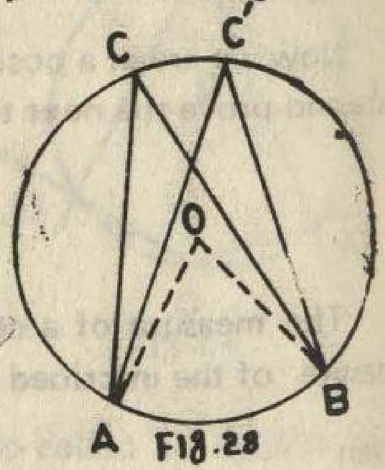


Fig.28

in figure 28, $\angle C$ and $\angle C'$ are two inscribed angles of a major arc ACB.

Since the measures of each of these angles is half of the measure of $\angle AOB$.

$\therefore m\angle C' = m\angle C$

COR. 2. An inscribed angle of a major arc is an acute angle.

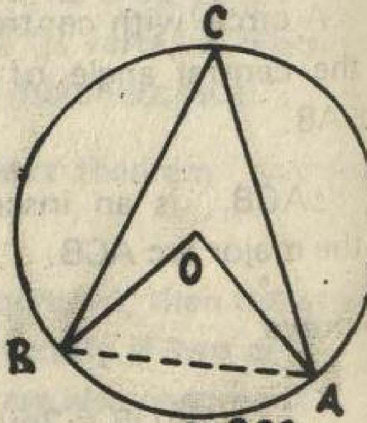


FIG.29

In figure (29),

$m\angle AOB < 180^\circ$

(The sum of the measure of all the three angles of a triangle is 180°. Therefore, the measure of each angle is less than 180°).

[$\frac{1}{2}m\angle AOB < 90^\circ$

Therefore,

$m\angle C < 90^\circ$

THEOREM 6

Any inscribed angle of a semi-circular arc is a right angle.

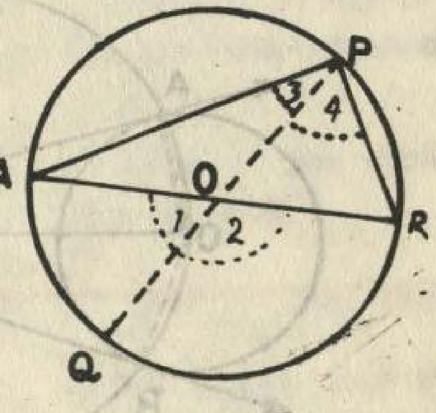


Fig. 30

Given

A circle with centre O.

APB is an inscribed angle of the semi-circle APB.

To Prove

$m\angle APB = 90^\circ$

Construction

Joint P to O and produce $\overrightarrow{PO}$ such that it meets the circle again at a point Q.

Proof

AQ is a minor arc, ∠ 1 is its central angle and ∠3 is its inscribed angle of the corresponding major arc.

$\therefore \quad m\angle 3 = \frac{1}{2}\, m\angle 1 \quad \ldots \quad$ (i)

Similarly, $m\angle 4 = \frac{1}{2}\, m\angle 2 \quad \ldots \quad$ (ii)

Adding the respective sides of (i) and (ii),

$m\angle P = \frac{1}{2}\,(m\angle 1 + m\angle 2)$

$= (\frac{1}{2} \times 180)^\circ$ (∠1 and ∠2 are adjacent supplementary angles)

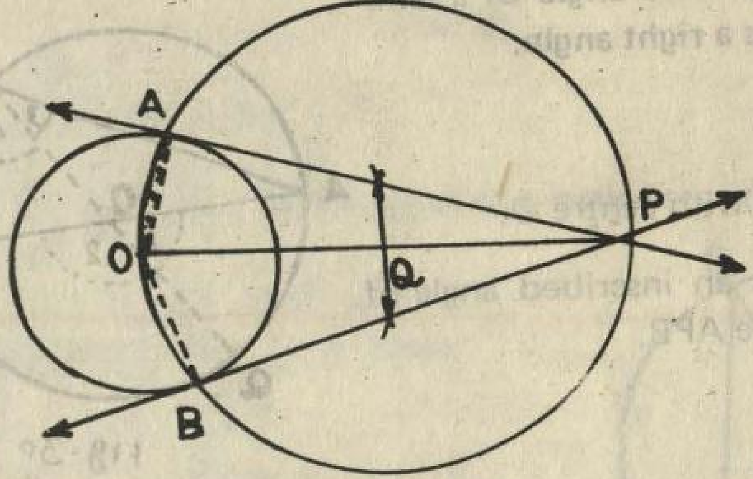
$= 90^\circ$

COR. All the three vertices of a right-angled triangle are equidistant from the mid-point of its hypotenuse.

N.B.: The phrase 'angle inscribed in a semi-circular arc' is, for convenience, shortenend as 'angle in a semi-circle'.

CONSTRUCTION THEOREM 2

To draw tangents to a circle from an exterior point.



Fig·31

Given

A circle with centre O and P an exterior point.

Required:

To draw the tangents from P to the circle.

Construction

With the help of a ruler join P to O. With the help of a ruler and compasses bisect $\overline{OP}$ at Q. With Q as centre and radius equal $\overline{PQ}$ (or $\overline{OQ}$) draw a circle intersecting the given circle in points A and B. Joint P to A and B and produce each of $\overleftrightarrow{PA}$, $\overleftrightarrow{PB}$ in both directions.

$\overleftrightarrow{PA}$, and $\overleftrightarrow{PB}$ are the required tangents,

N.B.: Since ∠ OAP is an angle in a semi-circle, it is a right angle,

Thus $\overline{OA} \perp \overleftrightarrow{AP}$

Therefore, $\overleftrightarrow{AP}$ is a tangent to the circle.

Similarly, $\overleftrightarrow{PB}$ is another tangent.

Definition

Each of the segments $\overline{PA}$, $\overline{PB}$ is called a TANGENT SEGMENT.

EXERCISE NO.11.9

N.B.: Wherever necessary, use Pythagoras theorem.

1. Draw a circle of 6cm. radius. Construct a regular* hexagon whose vertices lie on the circle. Measure the length of each of its sides.

[*Hint* : Construct six angles at the centre of the circle, adjacent to one another, measure of each angle being 60°.]

2. If three points A, B, C, divide a circle into three congruent parts, $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CA}$ what will be the measure of the central angle of each arc ?

3. If eight points lie in order on a circle and are the end-points of congruent arcs, what is the measure of the central angle of each arc ?

4. The radius of a circle is 5cm. A chord is at a distance of 4cm. from the centre. Find the length of the chord.

5. The radius of a circle is 13 cm. One of its chords is 24 cm. long. Find the distance of the chord from the centre.

6. Draw a circle of 5cm. radius. Take any point A inside the circle. Construct the chord of the circle which has A as its mid-point.

7. Take any three non-collinear points and draw circle passing through them.

8. Construct a triangle the lengths of whose sides are 4cm. 7cm. and 8cm. Draw a circle passing through its vertices.

9. Prove that if two chords of a circle are equidistant from the centre, they are congruent.

---

* A polygon having six sides is called a HEXAGON. A regular polygon is a polygon all of whose sides are of equal length.

10. Draw a circle of 4.5 cm. radius. Take any point A on it and draw a tangent to the circle at this point.

11. Draw a circle with radius equal to 5 cm. Draw a chord of the circle which is 8 cm. long. Construct the tangents at the end-points of the chord.

12. Draw a circle of 4.3 cm. radius. Consturct tangents to the circle from an exterior point.

13. Draw a circle of 6 cm. radius. Take a point P at a distance of 10 cm. from its centre. Construct the tangents to the circle from P. Measure each tangent segment and verify the measurements by Pythagoras Theorem

14. The radius of a circle is 7 cm. and the tangent segment from an exterior point is 9 cm. long. How far is that point from the centre of the circle ?

3. LENGTH OF AN ARC AND AREA OF A SECTOR OF A CIRCLE

Sector

You know that the union of a circle and its interior is called a CIRCULAR REGION. In figure (33) is shown a sub-set of a circular region. Its boundry consists of an arc AB and the radial segments OA, OB through its end-points. This is a SECTOR of the circle (more accurately, of the circular region).

A sector is a sub-set of a circular region which is bounded by an arc and the radial segments passing through its end-points.

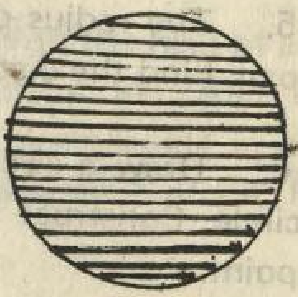


FIg.32

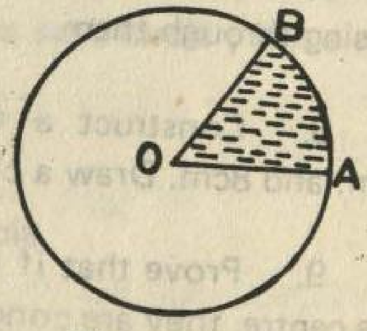


FIg.33

You also know that the length of a circle is called its CIRCUMFERENCE. If the radius of a circle is r, its circumference $= 2\pi r$

Here π is an irrational number whose approximate value is 3.14159.

Also if r is the radius of a circle, its area $= \pi r^2$

By using these two formulae we shall be able to find, quite easily, the length of an arc and the area of a sector.

A diameter of a circle divides it into two congruent parts. It also divides the region bounded by a circle into two congruent parts.

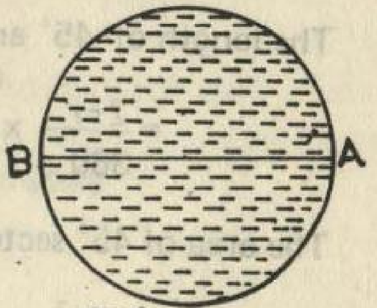


FIG: 34

$\therefore$ length of a semi-circle

$$= \frac{2\pi r}{2} = \pi r$$

and area of a semi-circular region $= \dfrac{\pi r^2}{2}$ (r being the radius of the circular region).

If two diameters $\overline{AB}$, $\overline{CD}$ of a circle are perpendicular to each other, then they will divide the circumference into four congruent arcs and the circular region will be divided into four congruent quadrants, which are known as 'quadrant circular regions."

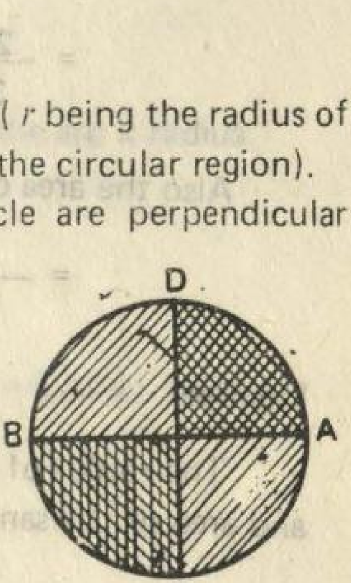


Fig. 35

Every such arc subtends 90° angle at the centre.

$\therefore$ Length of the quadrant circumference

$$\frac{2\pi r}{4} = \frac{\pi r}{2}$$

(*Note.*: $\frac{2\pi r}{4} = \frac{2\pi r}{360} \times 90$).

The area of 'Quardant Circular Region' (or 90° sector)

$$= \frac{\pi r^2}{4} \left(\frac{\pi r^2}{4} = \frac{\pi r^2}{360} \times 90 \right)$$

The arc which subtends x° angle at the centre, for convenience we call it x° arc, and the relevant sector is called x° sector.

So it is evident from the above-mentioned examples that :

The length of 45° arc

$$= \frac{2\pi r}{360} \times 45 = \frac{\pi r}{4}$$

The area of 45° sector

$$= \frac{\pi r^2}{360} \times 45 = \frac{\pi r^2}{8}$$

Similarly, the length of x° arc

$$= \frac{2\pi r}{360} \times x^\circ$$

Also the area of x° sector

$$= \frac{\pi r^2}{360} \times x^\circ$$

Example

The radius of a circle is 6 cm. Find the length of its 45° arc and area of the same degrees sector.

Solution

The length of 45° arc

$$= \frac{2\pi r}{360} \times 45 \text{ cm.}$$

$$= \frac{2\pi \times 6}{360} \times 45 \text{ cm.} = \frac{3\pi}{2} \text{ cm.}$$

$$= \frac{3}{2} \times 3.14 \text{ cm.} = 4.7 \text{ cm. nearly.}$$

Area of 45° sector

$$= \frac{\pi r^2}{360} \times 45 \text{ sq. cm.}$$

$$\frac{36\pi}{8} \text{ sq.cm.} = \frac{9\pi}{2} \text{ sq. cm.}$$

$$= \frac{9}{2} \times 13.14 \text{ sq. cm.}$$

$$= 14.1 \text{ sq. cm. approximately}$$

Example 2

The length of the arc of the sector of a circle is 42 meters. If the diameter of the circle is 60 meters, find the area of the sector.

Solution

Area of a sector = ½ x length of the arc x radius

Length of the arc = 42 meters

$$\text{radius} = \frac{60}{2} = 30 \text{ meters}$$

$\therefore$ Area of the sector = ½ x 42 x 30 = 630 sq. meters.

EXERCISE NO. 11.10

1. A circle is of 6cm. radius. Find the length of a 60° arc of this circle and also the area of the sector of the same angle.
2. The radius of a circular garden is 8 decameters. Find the area and perimeter of its 60° sector.

3. The area of a sector of a circle of 10m. radius is 40π Sq. m. Find the measure of the angle of the sector and the length of the corresponding arc.

4. Find the area of a 120° sector and the length of the corresponding arc of a circle of radius 9m.

5. The radius of each circle in the adjoining figure is 3 cm. Find the area of the shaded region.

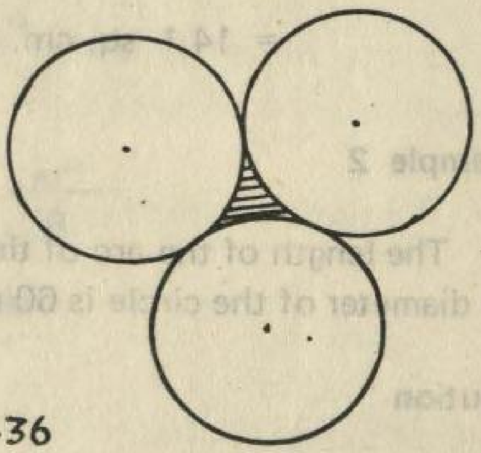


FIG.36

6. The radius of each circle in the adjoining figure is 4 cm. Find the area of the shaded region.

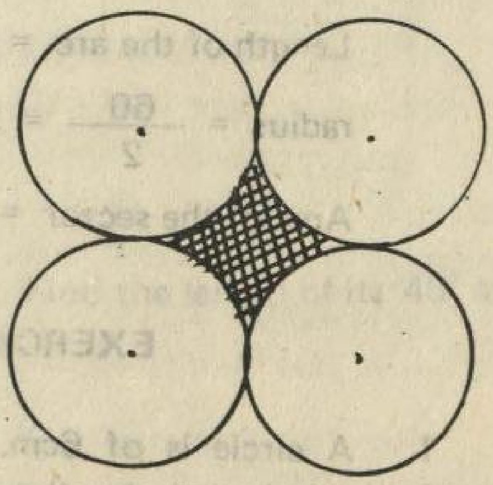


FIG.37

EXERCISE NO. 1.1

1. 17 2. 51 3. 375 4. 346 5. 698

6. $(43)_5$ 7. $(124)_5$ 8. $(301)_5$ 9. $(422)_5$ 10. $(2244)_5$

EXERCISE NO. 1.2

1. $(101)_2$ 2. $(10001)_2$ 3. $(100000)_2$

4. $(111111)_2$ 5. $(1010001)_2$ 6. 5

7. 14 8. 42 9. 121

10. 97

EXERCISE NO. 1.3

1. $(210)_5$ 2. $(1020)_5$ 3. $(123)_5$, 38

4. $(121)_5$, 36 5. $(143)_5$, 48 6. $(1103)_5$, 53

7. $(11)_5$ 8. $(234)_5$ 9. $(4)_5$

10. $(12)_5$ 11. $(124)_5$ 12. $(14)_5$

13. $(101)_5$ 14. $(212)_5$ 15. $(14)_5$

EXERCISE NO. 1.4

1. $(1041)_5$ 2. $(3324)_5$ 3. $(14230)_5$

4. $(32411)_5$ 5. $(234413)_5$

EXERCISE NO. 1.5

1. $(1000)_2$ 2. $(10011)_2$ 3. $(1110)_2$

4. $(1001)_2$ 5. $(1)_2$ 6. $(1000)_2$

7. $(1111)_2$ 8. $(1000001)_2$ 9. $(101111010)_2$

10. $(100)_2$ 11. $(11110100)_2$ 1 12. $(10100)_2$

EXERCISE NO. 2.1

(i) $A = \{x \mid x \in N, x \leq 100\}$ or $A = \{x \mid x \in N, 1 \leq x \leq 100\}$

(ii) $B = \{y \mid y$ is a side of $\triangle ABC\}$

(iii) $C = \{a \mid a \in I, -1 \geq a \geq -1000\}$

(iv) $D = \{a \mid a \in E, 2 \leq a \leq 5000\}$

(v) $E = \{x \mid x$ is an even integer $\}$

(vi) $O = \{x \mid x$ is an odd integer $\}$

(vii) $F = \{a \times b = 24 \mid a, b \in N\}$

(viii) $Q = \{x \mid x = \frac{a}{b}, a, b \in I, b \neq 0\}$

There can be so many other ways of writing the above sets in Set Builder Notation.

EXERCISE NO. 2.2

1. A, B, G, K, S, T, V are finite sets;
   C, D, E, F, H, J, L, M, N are infinite sets.

2. $P(a) = \{ \{2\}, \phi \}$, $P(B) = \{ \{ 1 \}, \phi \}$, $P(C) = \{ \{0\}, \phi \}$

$P(D) = \{ \phi \}$, $P(E) = \{ \{ 1 \}, \{ 2 \}, \{ 1, 2 \}, \phi \}$,

$P(F) = \{ \{ 0 \}, \{ 1 \}, \{ 0, 1 \}, \phi \}$,

$P(G) = \{ \{ -1 \}, \{ 1 \}, \{ -1, 1 \} \phi \}$

$P(H) = \{ \{ \frac{1}{2} \}, \{ -\frac{1}{2} \}, \{ \frac{1}{2}, -\frac{1}{2} \}, \phi \}$

$P(G_1) = \{ \{ \frac{1}{3} \}, \{ \frac{2}{5} \}, \{ \frac{1}{3}, \frac{2}{5} \}, \phi \}$

$P(H_1) = \{ \{ -\frac{1}{5} \}, \{ -\frac{5}{7} \}, \{ -\frac{1}{5}, -\frac{5}{7} \}, \phi \}$

$P(J) = \{ \{0\}, \{ 1 \}, \{ 2 \}, \{ 0, 1 \}, \{0, 2 \}, \{1, 2 \}, \{0,1,2 \}, \phi \}$

$P(K) = \{ \{ 2 \}, \{4 \}, \{6 \}, \{2,4 \}, \{2, 6\}, \{4, 6\}, \{2, 4, 6 \}, \phi \}$

$P(L) = \{ \{ -1 \}, \{ -2 \}, \{ -3 \}, \{ -1, -2 \}, \{ -1, -3 \}, \{ -2, -3 \},$
$\{ -1, -2, -3 \}, \phi \}$

$P(M) = \{ \{ \frac{1}{2} \}, \{ \frac{1}{4} \}, \{ 1/8 \}, \{ \frac{1}{2}, \frac{1}{4} \}, \{ \frac{1}{2}, 1/8 \}, \{ \frac{1}{2}, \frac{1}{4}, 1/8 \}, \phi \}$

$P(N) = \{ \{\sqrt{2}\}, \{ \sqrt{3} \}, \{ \sqrt{5} \}, \{ \sqrt{2}, \sqrt{3} \},$

$\{\sqrt{2}, \sqrt{5} \}, \{ \sqrt{3}, \sqrt{5} \}, \{ \sqrt{2}, \sqrt{3}, \sqrt{5} \}, \phi \}$

$P(S) = \{ \{2 \}, \{ \frac{1}{2} \}, \{\sqrt{2}\}, \{ 2, \frac{1}{2} \}, \{ 2, \sqrt{2} \}, \{ \frac{1}{2}, \sqrt{2} \},$

$\{2, \frac{1}{2}, \sqrt{2} \}, \phi \}$

$P(T) = \{ \{ 1 \}, \{ 2 \}, \{3 \}, \{4 \}, \{1, 2 \}, \{ 1, 3 \}, \{ 1, 4 \}, \{2, 3 \}$
$\{3, 4 \}, \{1, 2, 3 \}, \{1, 2, 4 \}, \{ 1, 3, 4 \}, \{2, 3, 4 \},$
$\{1, 2, 3, 4 \}, \phi \}$

$P(V) = \{ \{0 \}, \{ 1 \}, \{ -1\}, \{ -2 \}, \{0, 1 \}, \{0, -1 \}, \{0, -2 \},$
$\{1, -1 \}, \{ 1, -2 \}, \{ -1, -2 \}, \{0,1,-1\}, \{0, -1, -2 \},$
$\{ 1, -1, -2 \}, \{0, 1, -1, 2 \}, \phi \}$

$P(W) = \{ \{ 1/3 \}, \{ 2/3 \}, \{ -1/3 \}, \{ -2/3 \}, \{ 1/3, 2/3 \}, \{ 1/3, -1/3 \},$
$\{ 1/3, -2/3 \}, \{ 2/3, -1/3 \}, \{ 2/3, -2/3 \}, \{ -1/3, -2/3 \}$
$\{ 1/3, 2/3, -1/3 \}, \{ 1/3, 2/3, -2/3 \}, \{ 1/3, -1/3, -2/3 \},$
$\{ 2/3, -1/3, -2/3 \}, \{ 1/3, 2/3, -1/3, -2/3 \}, \phi \}$

$$P(X) = \{\{½\}, \{\sqrt{2}\}, \{0\}, \{2\}, \{½, \sqrt{2}\}, \{½, 0\},$$
$$\{½, 2\}, \{\sqrt{2}, 0\}, \{\sqrt{2}, 2\}, \{0, 2\},$$
$$\{½, \sqrt{2}, 0\}, \{½, 0, 2\} \{½, 0, 2\},$$
$$\{\sqrt{2}, 0, 2\}, \{½, \sqrt{2}, 0, 2\}, \phi\}$$

EXERCISE NO. 2.3

1. (i) to (iv) Yes, (v) No 2. (iii)

3. With respect to addition (i), (iii), (v), (vi)

With respect to multiplication : all sets except (iv)

With respect to subtraction : (i), (v),

With respect to division : none 8

EXERCISE NO. 2.4

(i) $(a, b) = (2, 3) \Leftrightarrow a = 2,\ b = 3.$

(ii) $(x, 3) = (-½, y) \Leftrightarrow x = -½,\ y = 3.$

(iii) $(x + 1, y + 2) = (5, 6) \Leftrightarrow x = 4,\ y = 4.$

2. (i) $A \times B = \{(1,4)(1,6),(1,8),(2,4),(2,6),2,8)\}$

$B \times A = \{(4,1),(4,2),(6,1),(6,2),(8,1),(8,2)\}$

(ii) $A \times B = \{(a,0),(a,-1),(b,0),(b,-1)\}$

$B \times A = \{(0,a),(0,b),(-1,a),(-1,b)\}$

(iii) $A \times B = \{(1,p),(1,q),(1,x),(1,y)\}$

$B \times B = \{(p,1),(q,1),(x,1),(y,1)\}$

3. (i) $A \times A = (1,1),(1,2),(2,1),(2,2)\}$

$B \times B = \{(4,4),(4,6),(4,8),(6,4),(6,6),$

$(6,8),(8,4),(8,6),(8,8)\}$

(ii) A X A = {(a,a),(a,b),(b,a),(b,b)

B X B = {(0,0),(0,−1),(−1,0),(−1,−1)}

(iii) A X A = {(1,1)}

B X B = (x,x),(x,y),(x,z),(x,s),(y,x),(y,y),(y,z),(y,
(y,s),(z ,x),(z,y),(z,z),(z,s),(s,x),(s,y),(s,z),(s,z)

5. I, I, II, IV, I, II, III, II, III, II, III, IV

6. y−axis, y−axis, x−axis, y−axis, y−axis,
y−axis, x−axis, x−axis, x−axis

EXERCISE NO. 3.1

1. 2^7 2. $(3)^{12}$ 3. $(x)^{12}$

4. $(\frac{1}{2})^9$ 5. b^3c^5 6. (x^{16})

7. 2^6 8. a^{12} 9. $(-3)^8$

10. $\left(\frac{-2}{7}\right)^{18}$. 11. 216 12. $\frac{1}{225}$

13. 144 14. a^8b^{12}

EXERCISE NO. 3.2

1. $\frac{27}{64}$ 2. $\frac{x^4}{y^4}$ 3. $\frac{-p^3}{q^3}$

4. $\frac{32a^5}{243b^5}$ 5. $\frac{a^4b^6}{c^8}$ 6. 3^6

7. a 8. a^6b^6 9. $(a+b)^4$

10. $(p+q)^{14}(r+s)^7$ 11. $\frac{1}{p^3}$ 12. $\frac{1}{b}$

13. $\frac{4a^3}{b^7}$ 14. $\frac{2b^4}{a^3}$ 15. $\frac{-2}{p^6q^2}$

EXERCISE NO. 3.3

1. $\frac{1}{5}$ 2 $\frac{2}{13}$ 3. $\frac{3}{8}$ 4. $|7|$

5. $\frac{3}{10}$ 6. $|xy|$ 7. Redicand 24, index 3

8. Redicand $\frac{2}{3}$, index 4 9. Redicand $\frac{5}{7}$, index 6

10. Redicand ab, index 5 11. Redicand $\frac{p}{q}$, index 3

12. $\frac{3b^2}{a^4}$ 13. $\frac{4y^3}{x^2}$ 14. $\frac{2x^4}{y^5z^6}$

15. $\frac{a^7}{b^5}$ 16. $\frac{8x^2}{y}$ 17. $p^{\frac{1}{9}}$ 18. $\frac{1}{5}$

19. 1.5×10^{10} 20. 5.7×10^7

21. 1.2×10^{-10} 22. 3.05×10^{-13}

23. 0.000,01034 24. 45600000

25. 0.000000000000009421 26. 4100000000000000000000

EXERCISES NO. 3.4

1. $6^2 = 36$ 2. $5^3 = 125$ 3. $10^2 = 100$

4. $a^0 = 1$ 5. $4^3 = 64$ 6. $\log_3 81 = 4$

7. $\log_5 625 = 4$ 8. $\log_2 \frac{1}{16} = -4$ 9. $\log_7 1 = 0$

10. $\log_{36} 6 = \frac{1}{2}$ 11. $\log_{10} 100000 = 5$

12. 2　13. 1　14. 6
15. 81　16. 4　17. 3
18. 9　19. 2　20. 5

EXERCISE NO. 3.5

1. $\log_a 5 + \log_a 7$　2. $\log_a 4 + \log_a 7 + \log_a 9$
3. $\log_a x + \log_a y$　4. $\log_a u + \log_a v + \log_a w$
5. $\log_a x - \log_a y$　6. $\log_a l + \log_a m - \log_a n$
7. $\log_a u + \log_a v - \log_a w - \log_a x$
8. $\log_a 21$　9. $\log_a xyz$　10. $\log_{10} \frac{12}{5}$
11. $\log_{10} 5^3 3^4$　12. 0.7781　13. 1176
14. 1.4771　15. 0.9031　16. 0.9542
17. 2.0970　18. 0.6505　19. 0.8807

EXERCISE NO. 3.6

1. 0.6990　2. 2.7396　3. 1.7589
4. 2.8972　5. 4.4455　6. 0.8727
7. 5.6599　8. 1.6740　9. 3.7782
10. 2.7565　11. 88　12. 277
13. 7521　14. 3.579　15. 0.0744
16. 0.005967　17. .01405

EXERCISE NO. 3.7

1. 284.5 2. 265.2 3. 45.42
4. 3.681 5. 3.212 6. 28.19
7. 38.60 8. 1968 9. 2.927
10. 11.20 11. 1.655 12. 1.848
13. 1.81 14. 4.745 15. 45.42
16. 23.81 17. 12.42 18. 5.716

EXERCISE NO. 4.1

1. (i) 5, (ii) 16, (iii) 11, (iv) 29, (v) 15, (vi) 45, (vii) 15, (viii) 61 – 75, (ix) 60
2. No. of Classes = 5
3. Class frequency of 52 – 53 = 9
4. Class interval with least frequency is 21 – 30
5. Interval with best attendance : 1 – 10
6. Number of workers earning Rs. 831 to Rs. 845 = 10
7. (i) False lower limit is 400
   (ii) False , upper limit 440 is not included
   (iii) True (iv) True
   (v) False, class-interval with least frequency is (400 – 420)

EXERCISE NO. 4.2

1. (i) 6, (ii) 32, (iii) 3007, (iv) 10.24
2. 58 3. 3

4. $\Sigma f = 52$, $x = 3, 8, 13, 18$, $\Sigma fx = 596$,
$D = -15, -10, -5, 0$
where provisional mean is 18
$\Sigma fD = -340$

5. Rs.18.83 (nearly)

6. (i) zero, (ii) 2, (iii) −15

7. 35 years 6 months

8. 34.3 9. Rs.358.51 (approx.) 10. 7.4 decimeter(approx.)

EXERCISE NO. 4.3

1. (i) 3, (ii) 3.5, (iii) 32, (iv) zero

2. 633 3. 27 4. 45.5 5. 3

6. Zero 7. 800 8. 55

MISCELLANEOUS EXERCISE NO. 4.4

2. 13.3 3. 320 4. 65

5. 660 6. A.M. = 70.48 , Mode = 68

7. 61.3 8. 354 years 2 months

9. Rs. 921.40

10. (i) True (ii) False, lower limit of 57 – 60 is 57
(iii) False, frequency of 60 – 63 is 125
(iv) False, central value of 51 – 54 is 52.5
(v) True
(vi) False, the number of students is 375.
(vii) True (viii) False, the number of students is 700
(ix) False, upper limit 63 is not included
(x) False, the class-interval with least frequency is 63 – 66

EXERCISE NO. 5.1

1. 5 : 6
2. 46 : 27
3. 7 : 10
4. 1 : 7
5. 3 : 4
6. Hameed
7. Hamid spent more
8. Nasim's expenditure is less
9. Progress
10. 30 : 31

EXERCISE NO. 5.2

1. (i) 4, (ii) 10
2. 20 spoons
3. 10 cm.
4. 7 hectograms
5. 108 km.
6. 15 kg
7. 20 m.
8. Rs. 44
9. 11 quintals
10. 6 hrs. 2 mins. 30 secs.

EXERCISE NO. 5.3

1. 160 men
2. 18 days
3. 100 men
4. 6¾ hours
5. 4½ hours
6. 3 days
7. 70 days
8. 10 days

EXERCISE NO. 5.4

1. Rs.15
2. Rs. 8.75
3. 484 masons
4. 22.5 kg.
5. Rs.2.25
6. 10 men
7. 20 days
8. 400 men
9. 6 men
10. Rs. 18

EXERCISE NO. 5.5

1. Saleem : Rs. 40, Kaleem : Rs. 50, Fahim : Rs.60
2. A : Rs. 11.25, B : Rs. 12.50, C : Rs. 8.75
3. A : Rs. 320, B : Rs.240, C : Rs. 192, D : Rs.224
4. Zaid : Rs. 420, Omar : Rs.450, Bakar : Rs. 360
5. 200, 240, 250 fishes
6. Tea water : 800 gm., milk : 100 gm., sugar : 150 gm.
7. Zafar : Rs.252, Mazhar : Rs.336, Anwar : Rs.448
8. A : Rs. 1380, B : Rs. 1840, C : Rs. 2208
9. Tahir : Rs. 125, Hamid : Rs. 150, Iqbal : Rs. 168. 75
10. A : Rs. 200, B : Rs. 300, C : Rs. 450, D : Rs. 600

EXERCISE NO. 5.6

1. Rs.160, Rs.210, Rs.260
2. Amjad : Rs. 35, Asghar : Rs.41, Asad : Rs. 50
3. Munir : Rs. 66, Bashir : Rs. 51, Nazir : Rs. 75
4. First share : 18, Second share : 27, Third share : 36
5. Ameena : Rs.45, Nasreen : Rs.30, Yasmeen Rs.120.
6. A : Rs. 500, B : Rs. 450, C : Rs.300
7. Hameed : Rs. 260, Waheed : Rs.275, Saeed : Rs.180
8. A : Rs. 48, B : Rs. 42, C : Rs. 22
9. Man : Rs. 40, Woman : Rs.20
10. A : Rs. 7.50, B : Rs. 6.25, C : Rs. 12.50
11. Boy : Rs. 25, Woman Rs. 40, Man : Rs.50
12. Officer's salary Rs 2000, Clerk's salary Rs.1000, Peaon's salary Rs.500

EXERCISE NO. 5.7

1. Mother : Rs.2500, Wife : Rs.1875, besides dower, Son : Rs.10625
2. Husband : Rs.675, each daughter : Rs.900, real brother : Rs. 225
3. Zaid : Rs.4500, Son : Rs.6750, Daughter : Rs.3375
4. Each son : Rs.4000, each daughter : Rs.2000
5. Wife : Rs.6000, mother : Rs.8000, father : Rs.8000, each daughter : Rs. 16000
6; Widow : Rs.2000, son : Rs.1750, daughter : Rs.875
7. Mother : Rs.6000, father : Rs.6000, each daughter : Rs.8000
8. Brother : Rs.8800, sister : Rs.4400
9. Daughter : Rs.5000, each son ; Rs.10,000

EXERCISE NO. 5.8

1. Zaid : Rs.420, Omar : Rs.450, Bakr : Rs.360
2. A : Rs.252, B : Rs.168, C : Rs.300
3. Karim : Rs.154 : Rahim : Rs.423.50, Saleem : Rs.308
4. Partner I : Rs.400, Partner II : Rs.360, Partner III : Rs.480
5. A : Rs.1920, B : Rs.2080, C : Rs. 2240
6. A : Rs. 640, B : Rs.600, C : Rs.280, D : Rs.520
7. A : Rs. 285, B : Rs.225, C : Rs.180

EXERCISE NO. 5.9

1. Rahim : Rs.64, Karim ; Rs.96
2. A : Rs.225, B : Rs.420, C : Rs.300

3. A : Rs.1500, B : Rs.1312.50, C : Rs.1187.50

4. A : Rs.96, B : Rs. 88, C : Rs.360

5 A : Rs. 880, B : Rs.1384, C : Rs.2000

6 A : Rs.736, B : Rs.736, C : Rs.552

7 3 months hence

8 7 months hence

EXERCISE NO. 6.1

1. 60%
2. 15%
3. 24%
4. 16%
5. 45.60
6. 400
7. 150
8. 32
9. Rs.120
10. Rs.5000

EXERCISE NO. 6.2

1. Rs.111.25
2. Rs.2260
3. **Rs.3840**
4. **Rs.11750**

Rs.15200

6. Rs.270
7. Rs. 22000
8. Rs. 495
9. Rs.39305
10. Rs.6750
11. Rs.4600
12. Rs.3400, Rs.850

EXERCISE NO. 6.3

1. Rs.90, Rs.360
2. Rs.32, Rs.128
3. Rs.760.50
4. Rs.64.60
5. Rs.994.50
6. 15%
7. Rs.189
8. Rs.4550
9. Rs.220, Rs.10890, Rs.11110,
10. Rs.1320

EXERCISE NO. 6.4

1. Rs.420 2. Rs.31.25 3. Rs.44, 16% $18\frac{2}{11}$%

4. Rs. 88, 32% 5. 32%, Rs.140

6. Rs.6000, Rs.5750, Rs.5500, Rs.5250

7. Rs.2600, Rs.2500, Rs.2400, Rs.2300, Rs.2200, Rs.2100

8. Rs.5375 , Rs.2750, Rs.2625

EXERCISE NO. 7.1

1. (i) $x^3 - 2x^2 + x - 1,\ -1 + x - 2x^2 + x^3$

   (ii) $-7x^5 + 8x^4 + x^3 + 6,\ 6 + x^3 + 8x^4 - 7x^5,$

   (iii) $-a^4 + 4a^2 - 5a - 3,\ 3 - 5a + 4a^2 - a^4,$

   (iv) $5x^6 - 2x^4 + 10x^2 + 4x + 3,\ 3 + 4x + 10x^2 - 2x^4 - 5x^6$

2. (i) $9x^3 - 5x^2 + 2x + 8$ (ii) $14a^2 - 6a + 2$

   (iii) $x^2y^2 + xy$

3. $-4a^5 - 10a^4 - 3a^2 + 8a - 9$

4. $x^4 - x^3 - x + 9$

5. (i) $3x + 5y$ (ii) $3a - 14b + c$

6. $3x^3 - 2x^2 - 14x - 6$ 7. $16x^4y - 6x^2y^2 + xy^3$

8. $5l^5 + 3l^4 + 7l^3 + 8l^2 - 7$ 9. $-6a + 11b - 5c$

10. $3x^3 - 11x^2 + x + 9$

EXERCISE NO. 7.2

1. $x^2 + 6x + 9$ 2. $x^2 - 81$ 3. $a^3 + b^3$

4. $8x^3 - 27y^3$ 5. $x^2 - y^2 - z^2 + 2yz$

6. $l^4 + l^2 + 1$ 7. $4a^6 + 10a^5 - 12a^4 - 17a^3 + 6a^2 + 8a - 3$

8. $a^3 + b^3 + c^3 - 3abc$ 9. $-14p - 25q - 4r$

10. $12x^3 - 11x^2 - 15x - 12$

EXERCISE NO. 7.3

1. $5x^2y$ 2. $11a^3$ 3. $2y^6z - 6y^5z^2 - 18y$

4. $x - 3$ 5. $x^3 + x^2y + xy^2 + y^3$ 6. $x^2 - x$

7. $x - 4$ 8. $-4x^2 - 1$ 9. $2 + 3y - 4y^2$

10. $a^2 + b^2 + c^2 - ab - bc - ca$ 11. $x^3 - 5x + 3$

12. $q = -12$

EXERCISE NO. 8.1

6. $3x^2y - 3xyz - 3x^2z$ 7. $10a^3 - 25a^2 + 5a$

8. $3x^2y - 2x^2z - 4x^5$ 9. $-12p + 8q - 16r$

10. $-5a^3b^2 - 110a^2b^3 + 15abc$ 11. $a^2 + ab + ca + bc$

12. $dm + nc$ 13. $-2abcd$

14. $x^2 + xy - x - y$ 15. $l^2m + lm^2 - l + m$

16. $a^2b^3c^2 + a^2b^2c^3 + a^3b^2c^2$

17. $3x^3y^2z^2 + 3x^2y^3z^2 - 3x^2y^2z^3$

18. $-20a^3k^2c + 12a^3k^3 - 8a^2k^2b$

19. $x^na^n + x^nb^n - x^nc^n$ 20. $a^2m - a^mb^n + a^mc^m$

EXERCISE NO. 8.2

11. $2a(2b - 4c - 3x)$ 12. $3xyz(2x + 3y + 4z)$

13. $5xy(x^2 + 3xy - 1)$ 14. $(p + q)(x + y)$

15. $(b - c)(ab + bc + ca)$ 16. $(a + b)(c + x)$

17. $(x + y)(a - 4)$ 18. $(b + 1)(a + c)$

19. $(a + c)(b + d)$ 20. $(b + x)(a + y)$

21. $(a - c)(3a - 5b)$ 22. $(x + y)(4 - ay)$

23. $(1 + a)(1 + b)$ 24. $b^x(a^x + b^x + b^y)$

25. (i) 1400   (ii) 235   (iii) 1700   (iv) 369

EXERCISE NO. 8.3

11. $16c^2 + 4c + \frac{1}{4}$ 12. $4x^2 - 12xy + 9y^2$

13. $4x^2 - 4 + \frac{1}{x^2}$ 14. $a^2 - 1 + \frac{1}{4a^2}$

15. $x^{2m} + 2x^m y^n + 2y^n$

16. $6a$ may be added.

17. y may be added or subtracted

18. $4x^2$ may be added or subtracted

19. c^2 may be added 20. b^2 may be added

EXERCISE NO. 8.4

1. 10201 2. 162409 3. 245025

4. 473344 5. 13 6. 85

7. 68 8. 41 9. 116

10. 58 11. 23 12. 51

13. 7 14. $m^2 + 2$ 15. 194

16. 7 17. 34 19. 20

20. 26

EXERCISE NO. 8.5

6. $(5 - c)^2$ 7. $(x + 4)^2$ 8. $(6x + 5y)^2$

9. $(a^2 - 3b^2)^2$ 10. $(5x^2y^2 - 1)^2$ 11. $(x^n + y^n)^2$

12. 100 13. 2500 14. 36

EXERCISE NO. 8.6

11. $m^4 - n^4$ 12. $16 - x^4$ 13. $x^8 - y^8$

14. $x^8 - 1$ 15. $x^{2m} - y^{2m}$ 16. $a^{2x} - b^{2x}$

17. 2 18. $a-b$ 19. 999375

20. 359900

EXERCISE NO. 8.7

11. $a(a+b)(a-b)$ 12. $xy(x+y)(x-y)$

13. $3(3+ab)(3-ab)$ 14. $(a^2+b^2)(a+b)(a-b)$

15. $(a^2+8)(a^2-8)$ 16. $3(x^2+4)(x+2)(x-2)$

17. $x(4x^2+1)(2x+1)(2x-1)$

18. $(x+y+z)(x+y-z)$

19. $(a-b+c)(a-b-c)$

20. (i) 270000 (ii) 1 (iii) 2.32 (iv) $\frac{1}{8}$

EXERCISE NO. 8.8

1. $(a+b+c)(a+b-c)$ 2. $(x-y+z)(x-y-z)$

3. $(a-1+b)(a-1-b)$ 4. $(a-x+b)(a-x-b)$

5. $(c+b+a)(c+b-a)$ 6. $(1+a-b)(1-a+b)$

7. $(b+c+a)(b-c-a)$ 8. $(2a+3x+b)(2a+3x-b)$

9. $(x^2+4x+8)(x^2-4x+8)$

10. $(2x^2+6x+9)(2x^2-6x+9)$

11. $(x^2+2xy+2y^2)(x^2-2xy+2y^2)$

12. $a(x^2+2x+2)(x^2-2x+2)$

13. $3(2a^2+2ab+b^2)(2a^2-2ab+b^2)$

14. $(a^2+ab+b^2)(a^2-ab+b^2)$

15. $(x^2+xy+y^2)(x^2-xy+y^2)$

16. $(a^2+a+1)(a^2-a+1)$

17. $(x^2+3+2x)(x^2+3-2x)$

18. $(x^2+2xy-y^2)(x^2-2xy-y^2)$

19. $(a^2+ab-3b^2)(a^2-ab-3b^2)$

20. $(a^2+5a-1)(a^2-5a-1)$

EXERCISE NO. 8.9

6. $x^2 + 7x - 60$ 7. $x^2 - 3x - 180$ 8. $x^2 - 15x - 54$

9. $x^2 - 4x - 5$ 10. $x^2 - 11x + 30$

EXERCISE NO.8.10

11. $(p + 6q)(p + 4q)$
12. $(x - 3)(x + 2)$
13, $(x - 12)(x - 5)$
14. $(x - 20)(x + 3)$
15. $(x - 28)(x + 3)$
16. $(x - 3)(x + 18)$
17. $2(x + 1)(x + 3)$
18. $3(x - 12)(x + 2)$
19. $xy(xy - 5)(xy - 4)$
20. $x(x + 6)(x - 4)$

EXERCISE NO. 8.11

5. $m^3 + 3m^2 n + 3mn^2 + n^3$

6. $8 + 12c + 6c^2 + c^3$

7. $27a^3 - 54a^2 b + 36ab^2 - 8b^3$

8. $64 - 96x + 48x^2 - 8x^3$

9. $x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$

10. $\frac{x^3}{y^3} - \frac{3x}{y} + \frac{3y}{x} - \frac{y^3}{x^3}$

11. 35 12. -35 13. 56 14. 98

15. 279 16. (i) 52, (ii) 18 17. (i) 76, (ii) -76

EXERCISE NO. 8.12

4. $(x + y)^2$ 5. $(p + 3)^3$ 6. $(x - 5y)^3$

7. $(1 - x)^3$ 8. $(x + \frac{1}{x})^3$ 9. $(m - \frac{1}{m})^3$

10. $(10 - xy)^3$

EXERCISE NO. 3.13

6. $8a^3 - 27$ 7. $a^6 - 1$ 8. $x^3 + \frac{1}{x^3}$

9. $\frac{x^3}{y^3} - \frac{y^3}{x^3}$ 10. $x^{3n} + y^{3n}$ 11. $a^{3m} - y^3$

12. $x^6 - y^6$ 13. $a^6 - b^6$ 14. $a^6 - 729$

15. $729a^6 - 64b^6$

EXERCISE NO. 3.14

6. $(3x + 4)(9x^2 - 12x + 16)$
7. $(x - yz)(x^2 + xyz + y^2z^2)$
8. $ab(a - b)(a^2 + ab + b^2)$
9. $a^2b^2(b - a)(b^2 + ab + b^2)$
10. $(x^2 + y^2)(x^4 - x^2y^2 + y^4)$
11. $(x^2 + 4y^2)(x^4 - 4x^2y^2 + 16y^4)$
12. $(x + y)(x - y)(x^2 + xy + y^2)(x^2 - xy + y^2)$
13. $(x + 1)(x - 1)(x^2 - x + 1)(x^2 + x + 1)$
14. $ab(a + b)(a - b)(a^2 + ab + b^2)(a^2 - ab + b^2)$
15. $(xy^2 - 4)(x^2y^4 + 4xy^2 + 16)$
16. $(a - b)(a^2 + ab + b^2)(a^6 + a^3b^3 + b^6)$
17. $(a + 3)(a - 3)(a^2 + 3a + 9)(a^2 - 3a + 9)$

18. 500 19. 400 20. 8

MISCELLANEOUS EXERCISE NO. 3.15

1. 20 2. 13 3. 34

4. 215 5. 117 6. 76

7. (i) 1010025 (ii) 487204 8. 9

9. 0.5 10. 8 11. 125

12. 0.25

13. $x(x^2 - xy + y^2)$ 14. $(x - y)(x - 3)$

15. $(1 + a^2)(1 + b^2)$ 16. $(a + b)(a + b + 3)$

17. $(ax - by)(bx - ay)$ 18. $(x^2 + y^2)(a^2 + b^2)$

19. $(a^2 + d^2)(b^2 + c^2)$ 20. $(5ab - 2c)^2$

21. $(a^x - b^y)^2$ 22. $(9a + 5)(9a - 5)$

23. $a(a + 3b)(a - 3b)$ 23. $(x^2 + 9)(x + 3)(x - 3)$

25. $(x^4 + y^4)(x^2 + y^2)(x + y)(x - y)$

26. $(a^{2x} + b^{2x})(a^x + b^x)(a^x - b^x)$

27. $(c - b + a)(c - b - a)$ 28. $(z + x - y)(z - x + y)$

29. $(4a + x + b)(4a + x - b)$

30. $(2x^2 + 2x + 1)(2x^2 - 2x + 1)$

31. $(a^2 + 4a + 8)(a^2 - 4a + 8)$

32. $(a^2 + 2ab + 9b^2)(a^2 - 2ab + 9b^2)$

33. $(x^2 + x + 9)(x^2 - x + 9)$

34. $(x^2 + xy - y^2)(x^2 - xy - y^2)$

35. $(8 + 3b - b^2)(8 - 3b - b^2)$

36. $ab(ab - 6)(ab - 4)$ 37. $(x - 2)(x + 1)$

38. $(a + 4b)(a - 3b)$ 39. $(a + 3b)((a^2 - 3ab + 9b^2)$

40. $x(x - 3)(x^2 + 3x + 9)$

41. $(x^m + y^m)(x^{2m} - x^m y^m + y^{2m})$

42. $(2a + 3)(2a - 3)(4a^2 + 6a + 9)(4a^2 - 6a + 9)$

43. $(x + 2)(x + 1)(x - 1)$ 44. $(x - 5)(x + 1)(x - 1)$

45. $(x - 1)(x + 2)(x^2 - 2x + 4)$

EXERCISE NO. 9.1

Solution sets of all the equations are infinite.

EXERCISE NO. 9.2

2. (v) and (vi) represent the same line.

(v) and (vii) are parallel lines.

(i) and (iii), (ii) and (iv) are parallel lines.

(ix) (1, 2) is common to the solution sets of both the equations. No other part is common.

(x) Yes.

(xi) Since the lines are parallel, their solution sets have no common pair.

(xii) Yes.

EXERCISE NO. 9.3

1. (i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 11 \\ 9 & 2 \end{bmatrix}$

2. (i) $\begin{bmatrix} 1 & \\ 2 & 8 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & 0 \\ 2 & 6 \end{bmatrix}$ (iv) $\begin{bmatrix} 4 & -2 \\ 0 & 6 \end{bmatrix}$

(v) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (vi) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(vii) $\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$ (viii) $\begin{bmatrix} 4 & -3 \\ 4 & 19 \end{bmatrix}$

3. A, C and E are equal matrices.

B and D are equal matrices.

4. $\begin{bmatrix} -3 & 8 \\ 5 & -9 \end{bmatrix}$

EXERCISE NO. 9.4

1. $AB = \begin{bmatrix} 17 & 13 \\ 12 & 13 \end{bmatrix}$, $BA = \begin{bmatrix} 11 & 16 \\ 9 & 19 \end{bmatrix}$

2. $CD = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $DC = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3. (i) $XY = \begin{bmatrix} -2 & -4 \\ 4 & 0 \end{bmatrix}$ (ii) $YX = \begin{bmatrix} -2 & 4 \\ -4 & 0 \end{bmatrix}$

(iii) $ZY = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ (iv) $YZ = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

(v) $XZ = \begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}$. (vi) $ZX = \begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}$

(vii) Yes, (viii) Yes.

EXERCISE NO. 9.5

1. (ii), (iii) and (v) are singular Matrices, whereas (i) and (iv) are Non-singular.

2. (i) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, (ii) $\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} -\frac{2}{7} & -\frac{1}{7} \\ \frac{3}{28} & -\frac{1}{14} \end{bmatrix}$

3. Statements , (iii) , (vi) , (viii) and (ix) True whereas (i), (iv), (v), (vii), and (x) and False.

4. $A^{-1} = \begin{bmatrix} \frac{10}{29} & -\frac{6}{29} \\ \frac{8}{29} & \frac{1}{29} \end{bmatrix}$ 5. $B^{-1} = \begin{bmatrix} \frac{\text{[illegible]}}{2} & \frac{3}{2} \\ \frac{1}{4} & \frac{5}{4} \end{bmatrix}$

EXERCISE NO. 9.6

1. (i) $\begin{bmatrix} 3 & 5 \\ 6 & 10 \end{bmatrix}$, (ii) $\begin{bmatrix} \frac{1}{2} & -3 \\ 1 & -6 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix}$, (iv) $\begin{bmatrix} 3 & -2 \\ 5 & 2 \end{bmatrix}$

(i) and (ii) are singular Matrices whereas (iii) and (iv) are Non-singular.

2. (i) and (iii) do not possess solution sets whereas (ii) and (iv) possess solution sets.

3. BA and CA are not possible.

4. (i) $\begin{bmatrix} 4 & -3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$

Find the answers of (ii) and (iii) yourself.

5. (i) $\{(1, 1)\}$, (ii) $\{(-\frac{7}{2}, \frac{5}{2})\}$ (iii) $\{(3, 4)\}$

(iv) $\{(5, -1)\}$, (v) $\{(-1, 2)\}$

EXERCISE NO. 9.7

1. Solution set = $\{(3, 1)\}$ 2. Solution set = $\{(\frac{3}{2}, 1)\}$

3. Solution set = $\{(3, 4)\}$ 4. Solution set = $\{(-\frac{1}{3}, -\frac{1}{4})\}$

5. Solution set = $\{(-1, 2)\}$ 6. Price of one football is Rs.40 and price of one cricket ball is Rs 20. Price of 4 footballs and 2 cricket balls = Rs.200. 7. 42 years, 12 years. 8. Rs. 16, Rs.14

9. $\frac{7}{12}$ 10. Length of the rectangle = 4 m.
Breadth of the rectangle = 2 m.

EXERCISE NO. 10.1

1. $\overleftrightarrow{EF}$, $\overline{CD}$, $\overrightarrow{AB}$ 3. $\overline{AB}$, $\overline{BD}$. $\overline{DE}$ $\overline{EA}$

4. $\overrightarrow{LA}$, $\overrightarrow{LB}$, $\overrightarrow{MA}$, $\overline{MB}$, $\overrightarrow{NA}$, $\overrightarrow{NB}$

EXERCISE NO. 10.2

1. (iii) ∠ LAC or ∠ ABL (iv) **∠ACD**

(v) (∠ ACL, ∠ ACD), (∠ALB, ∠ALC)

(vi) (∠ BAL , ∠ ACD)

(vii) The outer arms of adjacent supplementary angles are collinear.

2. Yes 3. No 4. 90° 5. 45° 7. (i) 180°

8. (i) (∠ BPD, ∠ APC), (∠ CPB , ∠APD)

(ii) (∠ DPA , ∠ DPB), (∠ BPD, ∠ BPC)
(∠ CPA , ∠ CPB), (∠ APD , ∠ APC)

EXERCISE NO. 10.3

(i) The outer arms of the angles are collinear.

(ii) $m\angle DBA + m\angle DBC = 180°$

(iii) Vertical angles are congruent and pairs of adjacent angles are supplementary.

3. $m\angle 4 = m\angle 4$ 4. 156° 7. 135°

EXERCISE NO. 10.4

1. Parallel lines are coplanar and do not intersect.
2. Two parallel line segments be on parallel line.
3. Yes
4. (i) similarly directed. (ii) oppositely directed.
   (iii) oppositely directed (iv) similarly directed.
   (v) similarly directed. (vi) opposite
5. No.

EXERCISE NO. 10.5

1. Two points.
2. (i) $(\angle 3, \angle 5)$, $(\angle 4, \angle 6)$
   (ii) $(\angle 1, \angle 7)$, $(\angle 2, \angle 8)$
   (iii) $(\angle 2, \angle 6)$, $(\angle 3, \angle 7)$, $(\angle 1, \angle 5)$, $(\angle 4, \angle 8)$
3. (i) lines l, m (iii) lines p, q
5. (i) Vertical angles, (ii) Adjacent angles.
   (iii) Corresponding angles, (iv) Alternate angles,
   (v) Exterior alternate angles.

EXERCISE NO. 10.6

1. (i) Line l will intersect the line CD also.
   (ii) $\overleftrightarrow{EF}$ is parallel to $\overleftrightarrow{AB}$.
   (iii) Alternate angles are congruent, corresponding angles are congruent and the interior angles on the same side of EF are supplementary.
   (iv) Lines AB and CD are parallel.

2. $m\angle 1 = m\angle 2$ 3. $m\angle 2 = m\angle 5 = m\angle 7 = 135^\circ$

4. $m\angle 1 = \angle 3 = m\angle 8 = m\angle 6 = 45^\circ$

6. $m\angle 1 = 50^\circ$, $m\angle 2 = 70^\circ$, $m\angle 3 = 110^\circ$, $m\angle 4 = 130^\circ$

EXERCISE NO. 10.7

2. All correspondences.

3. $\triangle PBC \longleftrightarrow \triangle PDA$ and $\triangle PAB \longleftrightarrow \triangle PCD$.

4. $\angle CAB \cong \angle CAD$, $\angle ACD \cong \angle ACB$, $\angle ABC \cong \angle ADC$.

$\overline{AB} \cong \overline{AD}$, $\overline{BC} \cong \overline{CD}$, $\overline{AC} \cong \overline{AC}$

EXERCISE NO. 10.8

2. No. 3. Congruent

EXERCISE NO. 11.1

1. (i) one, (ii) infinitely many, (iii) one, (iv) impossible, (v) one, (vi) one, (vii) one, (viii) one, (ix) None, (x) one. (xi) impossible , (xii) Impossible.

2. The three sides, or two sides and their included angle, or one side and any two angles.

EXERCISE NO. 11.2

1. (i) 10 cm. (ii) 7 units, (iii) 40 mm., (iv) $\sqrt{19}$ cm.,

3. B, C, D 4. (i) 8.5 m., (ii) 25 m., (iii) 130 m.

5. 2.9 m. 6. 60 m. 7. 45 dm., -

EXERCISE NO. 11.3

1. AB = $\sqrt{x^2+y^2}$, DF = $2x$, EF = $\sqrt{3x^2}$

HK = x, GH = $\sqrt{3x^2}$, LN = x, MN = $\sqrt{2x^2}$

2. Yes 3. 1 unit 4. Yes

5. 1:2 6. $6\sqrt{2}$

7. AC = 10 cm., AB = $5\sqrt{2}$ cm., CD = $5\sqrt{3}$ cm.,

BD = 5 cm.,

8. 160 miles to the North and 277.3 miles to the East.

9. $4\sqrt{3}$ cm. $16\sqrt{3}$ sq.cm.

EXERCISE NO. 11.4

1. (i) correct, (ii) incorrect, (iii) correct,

(iv) incorrect, (v) incorrect, (vi) correct,

(vii) correct (viii) correct

2. $\triangle$ABC and $\triangle$LMN, $\triangle$DEF and $\triangle$GHK

3. 4.5 cm., 7.5 cm. 4. Yes 7. All pairs are similar.

8. 78 dm.,

EXERCISE NO. 11.6

1. (i) 2, (ii) $-\frac{1}{3}$ (iii) zero (iv) $-\frac{2}{3}$

2. zero 3. No.

EXERCISE NO. 11.7

1. $3x+2y-7=0$ 2. $x-y+2=0$ 3. $y=0$

4. (i) $\frac{2}{5}$, (0, 3), (ii) $-\frac{5}{2}$, (3, −2), (iii) 1, (0, 0),

(iv) 3, (0, 0), (v) 1, (0, −6).

EXERCISE NO. 11.8

1. chord : $\overline{PQ}$, diameter : $\overline{LM}$, tangent : $\overleftrightarrow{AB}$
point of contact : C, semi-circle : $\overset{\frown}{LPM}$, $\overset{\frown}{LCM}$.

3. Neither a minor arc nor a major arc.

4. Yes, no. 5. Only one, infinitely many

EXERCISE NO. 11.9

1. Length of a side : 6 cm. 2. 120° 3. 45°

4. 6 cm. 5. 5 cm. 13. 8 cm. 14. 11.4 cm. (approx.)

EXERCISE NO. 11.10

1. 6.3 cm. (approx.), 18.8 sq. cm.
2. 33.5 sq. decameter, 24.4 decameter (approx.)
3. 144°, 25.12 meter
4. 84.8 sq.m., 18.8 m.
5. Intersection : $(96\pi - 72\sqrt{3})$ sq. cm.
Union : $(192\pi + 72\sqrt{3})$ sq. cm.
6. $9(\sqrt{3} - \frac{\pi}{2})$ sq.cm.
7. $(64 - 16\pi)$ sq. cm.

The Quaid Said,

"You must devote yourself whole-heartedly to your studies, for that is your first obligation to yourselves, your parents and to the state."

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3	3 4 5	6 7 8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1 2 3	3 4 5	6 7 8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 2 2	3 4 5	6 7 7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1 2 2	3 4 5	6 6 7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1 2 2	3 4 5	6 6 7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1 2 2	3 4 5	5 6 7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1 2 2	3 4 5	5 6 7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1 2 2	3 4 5	5 6 7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1 1 2	3 4 4	5 6 7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1 1 2	3 4 4	5 6 7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1 1 2	3 4 4	5 6 6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1 1 2	3 4 4	5 6 6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1 1 2	3 3 4	5 6 6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1 1 2	3 3 4	5 5 6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1 1 2	3 3 4	5 5 6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1 1 2	3 3 4	5 5 6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1 1 2	3 3 4	5 5 6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1 1 2	3 3 4	5 5 6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1 1 2	3 3 4	4 5 6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1 1 2	2 3 4	4 5 6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1 1 2	2 3 4	4 5 6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1 1 2	2 3 4	4 5 5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1 1 2	2 3 4	4 5 5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1 1 2	2 3 4	4 5 5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1 1 2	2 3 4	4 5 5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1 1 2	2 3 3	4 5 5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1 2	2 3 3	4 5 5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1 1 2	2 3 3	4 4 5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1 1 2	2 3 3	4 4 5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1 1 2	2 3 3	4 4 5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1 1 2	2 3 3	4 4 5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1 1 2	2 3 3	4 4 5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1 1 2	2 3 3	4 4 5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1 1 2	2 3 3	4 4 5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1 1 2	2 3 3	4 4 5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1 1 2	2 3 3	4 4 5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1 1 2	2 3 3	4 4 5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0 1 1	2 2 3	3 4 4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0 1 1	2 2 3	3 4 4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0 1 1	2 2 3	3 4 4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0 1 1	2 2 3	3 4 4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0 1 1	2 2 3	3 4 4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0 1 1	2 2 3	3 4 4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0 1 1	2 2 3	3 4 4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0 1 1	2 2 3	3 4 4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0 1 1	2 2 3	3 4 4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0 1 1	2 2 3	3 4 4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0 1 1	2 2 3	3 4 4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0 1 1	2 2 3	3 4 4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0 1 1	2 2 3	3 3 4

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
10	0000	0043	0086	0128	0170						5 9 13	17 21 26	30 34 38
						0212	0253	0294	0334	0374	4 8 12	16 20 24	28 32 36
11	0414	0453	0492	0531	0569						4 8 12	16 20 23	27 31 35
						0607	0645	0682	0719	0755	4 7 11	15 18 22	26 29 33
12	0792	0828	0864	0899	0934						3 7 11	14 18 21	25 28 32
						0969	1004	1038	1072	1106	3 7 10	14 17 20	24 27 31
13	1139	1173	1206	1239	1271						3 6 10	13 16 19	23 26 29
						1303	1335	1367	1399	1430	3 7 10	13 16 19	22 25 29
14	1461	1492	1523	1553	1584						3 6 9	12 15 19	22 25 28
						1614	1644	1673	1703	1732	3 6 9	12 14 17	20 23 26
15	1761	1790	1818	1847	1875						3 6 9	11 14 17	20 23 26
						1903	1931	1959	1987	2014	3 6 8	11 14 17	19 22 25
16	2041	2068	2095	2122	2148						3 6 8	11 14 16	19 22 24
						2175	2201	2227	2253	2279	3 5 8	10 13 16	18 21 23
17	2304	2330	2355	2380	2405						3 5 8	10 13 15	18 20 23
						2430	2455	2480	2504	2529	3 5 8	10 12 15	17 20 22
18	2553	2577	2601	2625	2648						2 5 7	9 12 14	17 19 21
						2672	2695	2718	2742	2765	2 4 7	9 11 14	16 18 21
19	2788	2810	2833	2856	2878						2 4 7	9 11 13	16 18 20
						2900	2923	2945	2967	2989	2 4 6	8 11 13	15 17 19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2 4 6	8 11 13	15 17 19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2 4 6	8 10 12	14 16 18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2 4 6	8 10 12	14 15 17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2 4 6	7 9 11	13 15 17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2 4 5	7 9 11	12 14 16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2 3 5	7 9 10	12 14 15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2 3 5	7 8 10	11 13 15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2 3 5	6 8 9	11 13 14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2 3 5	6 8 9	11 12 14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1 3 4	6 7 9	10 12 13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1 3 4	6 7 9	10 11 13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1 3 4	6 7 8	10 11 12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1 3 4	5 7 8	9 11 12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1 3 4	5 6 8	9 10 12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1 3 4	5 6 8	9 10 11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1 2 4	5 6 7	9 10 11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1 2 4	5 6 7	8 10 11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1 2 3	5 6 7	8 9 10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1 2 3	5 6 7	8 9 10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1 2 3	4 5 7	8 9 10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1 2 3	4 5 6	8 9 10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1 2 3	4 5 6	7 8 9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1 2 3	4 5 6	7 8 9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1 2 3	4 5 6	7 8 9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1 2 3	4 5 6	7 8 9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1 2 3	4 5 6	7 8 9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1 2 3	4 5 6	7 7 8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1 2 3	4 5 5	6 7 8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1 2 3	4 4 5	6 7 8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1 2 3	4 4 5	6 7 8

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